61768

1913-1914

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

ORMOND STONE
L. P. EISENHART
OSWALD VEBLEN
MAXIME BÔCHER
ELIJAH SWIFT
J. H. M. WEDDERBURN

PUBLISHED UNDER THE AUSPICES OF PRINCETON UNIVERSITY

SECOND SERIES, VOL. 15.

LANCASTER, PA., AND PRINCETON, N. J.

SINGULAR POINT TRANSFORMATIONS IN TWO COMPLEX VARIABLES.*

By Guy Roger Clements.

This paper will be devoted to a detailed study of the transformation
T: $x = f(u, v)$, $y = \varphi(u, v)$,

(a) $f(u, v)$ *and* $\varphi(u, v)$ *denoting functions of the complex variables* u *and* v, *singlevalued and analytic throughout a neighborhood* R *of* $u = 0$, $v = 0$;

(b) $f(0, 0) = 0, \quad \varphi(0, 0) = 0$;

(c) $J(u, v) \equiv f_u\varphi_v - f_v\varphi_u \not\equiv 0, \quad J(0, 0) = 0$.

To any point (u, v) of R there corresponds, under T, one and but one point (x, y) of the xy-space. The totality of points (x, y) which thus correspond to points (u, v) of R constitute a finite region $\bar{R}$ of the xy-space. In general more than one point (u, v) will yield the same pair of values for x and y. We shall count a point (x, y) but once, thus regarding it as completely characterized by its coördinates, and shall seek an inverse transformation

$$u = \bar{f}(x, y), \qquad v = \bar{\varphi}(x, y),$$

which will put in evidence all those points (u, v) of R which correspond under T to any point of $\bar{R}$ with coördinates (x, y). Thus the functions $\bar{f}(x, y)$ and $\bar{\varphi}(x, y)$ are to be defined only in the points of $\bar{R}$ and there as singlevalued or as multiple-valued functions.

We shall distinguish two types of neighborhood.

If $x_1, x_2, \cdots, x_n$ denote n independent complex variables, any region which constitutes at least the totality of all points $(x_1, x_2, \cdots, x_n)$ for which

$$|x_1 - a_1| < \epsilon, \qquad |x_2 - a_2| < \epsilon, \cdots, \qquad |x_n - a_n| < \epsilon,$$

for some $\epsilon > 0$, is called a *complete* neighborhood of the point $(a_1, a_2, \cdots, a_n)$.

Any region which satisfies the conditions that

(1) for every $\epsilon > 0$, it contains points $(x_1, x_2, \cdots, x_n)$ for which

$$|x_1 - a_1| < \epsilon, \qquad |x_2 - a_2| < \epsilon, \qquad \cdots, \qquad |x_n - a_n| < \epsilon,$$

(2) for no $\epsilon > 0$ is every point $(x_1, x_2, \cdots, x_n)$ for which

$$|x_1 - a_1| < \epsilon, \qquad |x_2 - a_2| < \epsilon, \qquad \cdots, \qquad |x_n - a_n| < \epsilon,$$

a point of the region,

is called a *partial* neighborhood of the point $(a_1, a_2, \cdots, a_n)$.

* The results of this paper were in part announced in the Bulletin of the American Mathematical Society, 2d Series, Vol. XVIII (1912), pp. 454–456.

For purposes of simplicity the neighborhood of the point $(0, 0, \cdots, 0)$ is considered throughout this paper. In what follows the unqualified word *neighborhood* is used to mean *complete neighborhood*. R of the transformation T is a complete neighborhood.

2. Factoring.

A transformation can often be factored to advantage, that is, be replaced by the succession of two or more transformations each of which is simpler or better understood than the given one. We illustrate with a very useful example:

$$t: \qquad x = h^m, \qquad y = h^n(g^p + c),$$

where $h \equiv h(u, v)$, $g \equiv g(u, v)$ are single-valued and analytic functions of u and v in the point $(0, 0)$ and vanish there, where m, n, p are positive integers and where c is a constant. t is equivalent to the $n + 4$ transformations:

$$\begin{array}{rlrl}
u_0 &= h(u, v), & v_0 &= g(u, v), \\
u_1 &= u_0, & v_1 &= v_0^p, \\
u_2 &= u_1, & v_2 &= u_1 v_1, \\
\cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
u_{n+1} &= u_n, & v_{n+1} &= u_n v_n, \\
u_{n+2} &= u_{n+1}, & v_{n+2} &= v_{n+1} + c u_{n+1}^n \\
x &= u_{n+2}^m, & y &= v_{n+2}.
\end{array}$$

This makes use of the two types

$$r: \ x = u, \ y = uv; \qquad \text{and} \qquad s: \ x = u, \ y = v^n,$$

which we shall discuss in detail. We note that each is wholly independent of the coefficients of t. The only other factor which is not surely one-to-one and analytic both ways is

(1) $$u_0 = h(u, v), \qquad v_0 = g(u, v).$$

We can retrace our steps, build one inverse on another and thus get a picture of u_0 and v_0 as functions of x and y, and of the relations between their regions of definition. If the inverse of (1) can be found, the entire inverse can be built up and the solution is complete.

We examine the auxiliary transformations r and s.

$$r: \qquad x = u, \qquad y = uv; \qquad J = u.$$

For definiteness in the geometric image consider first the special case

that u and v have only real values. Let R be the region bounded by the lines $u = \pm 1$, $v = \pm 1$. The corresponding region $\bar{R}$ in the xy-plane is bounded by the lines $x = \pm y$, $x = \pm 1$; any line $v = au$ goes over into the parabola $y = ax^2$. To the locus $u = 0$ corresponds the point $x = 0$, $y = 0$ and conversely. That is, the inverse of r explodes the point $x = 0$, $y = 0$ into the line segment $u = 0$ of R. There is no point in $\bar{R}$ for which $x = 0$, $y \neq 0$. To every point of $\bar{R}$ other than $(0, 0)$ there corresponds one and but one point (u, v) of R determined by the relations $u = x$, $v = y/x$. The right members of these equations are single-valued and analytic at every point of $\bar{R}$ except at the origin where we have seen geometrically what is happening.

For the less simple transformation

$$r': \qquad x = u, \qquad y = u^n v, \qquad J = u^n,$$

the discussion is the same except that $\bar{R}$ is bounded by the curves $y = \pm x^n$, $x = \pm 1$. Further, r' is equivalent to r repeated n times.

To see the analogous results when u and v are allowed complex values let the definition of R be $|u| < 1$, $|v| < 1$. Then since $|x| = |u|$, $|y| = |u| \cdot |v|$, it is evident that the circles $|u| = 1$, $|v| = 1$ go over into the circles $|x| = 1$, $|y| = 1$, and that if a point in the x-plane be marked inside this circle $|x| = 1$, then a corresponding point y must be chosen for which y is less in absolute value that this x if the resultant point (x, y) is to be a point of $\bar{R}$. This shows that $\bar{R}$ is only a partial neighborhood of the origin. It follows as for the case of real variables above that the inverse is well defined, single-valued and analytic in every point of $\bar{R}$ except the origin and that the origin has an infinitely many-valued inverse $u = 0$, $v = v$.

$$s: \qquad x = u, \qquad y = v^n, \qquad J = nv^{n-1}.$$

Let R be defined by $|u| < u_0 > 0$, $|v| < v_0 > 0$. Take any point (u', v') of R and let (x', y') denote its image in $\bar{R}$. There are n points (u, v) of R which have the same image (x', y'), namely those for which $u = u'$, v equals the product of v' by any one of the nth roots of unity. Hence conversely to a point of $\bar{R}$ with coördinates (x', y') there correspond n distinct points (u, v) of R, the relationship being defined by the equations

$$u = x, \qquad v = y^{1/n}.$$

$y^{1/n}$ is an n-valued continuous function of x and y whose branches come together in $x = x$, $y = 0$.

Since we shall use power series freely, the definition $|u| < u_0$, $|v| < v_0$ for R is the one of practical use for this paper. For the problem immediately in hand however, we might have taken the u-space arbitrarily and for the

v-space any region which goes over into itself when rotated about $v = 0$ through an angle $2\pi/n$. For any other region the inverse would be in part n-fold, in part less than n-fold.

3. The Factoring of T.

Subject to a suitable choice of notation, every transformation T is of one of the two following mutually exclusive types:

(A) neither $f(u, v)$ nor $\varphi(u, v)$ is a factor of the other in the point $(0, 0)$;

(B) $\varphi(u, v) = f(u, v)^n[C + g(u, v)]$, where n is a positive integer, where C is a constant, and where $g(u, v)$ is a single-valued and analytic function of u and v in the point $(0, 0)$ and vanishes there.

We shall consider any transformation T to be *completely factored* when it is replaced by a transformation of type A combined with those which are one-to-one and analytic both ways, and with r and s of section 2.

Type B is the special case of the transformation t for which $m = 1$, $p = 1$. Consequently a first factoring is:

$$
\begin{array}{lll}
a: & u_0 = f(u, v), & v_0 = g(u, v); \\
b_1: & u_1 = u_0, & v_1 = u_0 v_0; \\
\cdot & \cdots\cdots & \cdots\cdots \\
b_n: & u_n = u_{n-1}, & v_n = u_{n-1}v_{n-1}; \\
c: & x = u_n, & y = v_n + Cu_n^n.
\end{array}
$$

c is one-to-one and analytic both ways; $b_1, \cdots, b_n$ are the transformations r of the preceding section; if a is one-to-one and analytic both ways or is of type A, the factoring is complete.

If a is again of type B the process must be repeated. We show that *complete factoring* is always attainable in a finite number of steps. To see this, we consider the series form for $f(u, v)$ and for $\varphi(u, v)$, first finding a normal form for $f(u, v)$. Let

$$f(u, v) = a_0(u) + a_1(u)v + a_2(u)v^2 + \cdots,$$

where we may assume without essential restriction that $a_0(u)$ is not identically zero, since we could always secure this by a proper linear transformation on u and v. Suppose

$$a_0(u) = b_m u^m + b_{m+1}u^{m+1} + \cdots, \quad b_m \neq 0,$$

and make the change of variable

$$u_1 = u\sqrt[m]{b_m + b_{m+1}u + \cdots}, \quad v_1 = v,$$

a transformation one-to-one and analytic both ways. We denote its inverse

by $u = h(u_1)$, $v = v_1$, whence

$$f[h(u_1), v_1] = u_1^m + A_1(u_1)v_1 + A_2(u_1)v_1^2 + \cdots.$$

We assume that $f(u, v)$ has this normalized form and consequently have before us

$$f(u, v) = u^m + A_1(u)v + A_2(u)v^2 + \cdots, \tag{2}$$

$$\varphi(u, v) = B_0(u) + B_1(u)v + B_2(u)v^2 + \cdots, \tag{3}$$

where

$$B_0(u) = b_p u^p + b_{p+1}u^{p+1} + \cdots. \tag{4}$$

Perform as many times as possible the process of factoring powers of $f(u, v)$ out of $\varphi(u, v)$ as indicated in the identity:

$$\varphi(u, v) = f^{n_1}\{C_1 + f^{n_2}[C_2 + \cdots + f^{n_k}(C_k + g_k(u, v)) \cdots]\} \tag{5}$$

$$= C_1 f^{n_1} + C_2 f^{n_1+n_2} + \cdots + C_k f^{n_1+\cdots n_k} + f^{n_1+\cdots+n_k} g_k(u, v)$$

$$= F(f) + f^N g_k(u, v), \tag{6}$$

where $C_1, C_2, \cdots, C_k$ are constants which, with the possible exception of C_k, are different from zero, where $n_1, n_2, \cdots, n_k$ are positive integers whose sum is N, where $F(f)$ is a polynomial in f of degree not greater than N, and where $g_k(0, 0) = 0$, but $f(u, v)$ is not a factor of $g_k(u, v)$ in the point $(0, 0)$.

To compute the coefficients C_i of F, assume for $g_k(u, v)$ a formal expansion $d_0(u) + d_1(u)v + d_2(u)v^2 + \cdots$, and combine this with (2) and (5) to derive

$$\varphi(u, v) = C_1[u^m + A_1(u)v + \cdots]^{n_1} + \cdots + C_k[u^m + A_1(u)v + \cdots]^{n_1+\cdots+n_k} \tag{7}$$

$$+ [u^m + A_1(u)v + \cdots]^{n_1+\cdots+n_k}[d_0(u) + d_1(u)v + \cdots]$$

$$= C_1 u^{mn_1} + C_2 u^{m(n_1+n_2)} + \cdots + C_k u^{m(n_1+\cdots+n_k)} + \text{higher powers of } u + \text{terms involving } v \text{ also.}$$

Compare (3) with (7) and equate the coefficients of like powers of u in the terms involving u alone. This gives

$$C_i = b_{m(n_1+n_2+\cdots+n_i)}, \quad i = 1, 2, \cdots, k. \tag{8}$$

But $\sum_{j=1}^{\infty} b_j u^j$ is absolutely and uniformly convergent, and consequently, for any N, $F(f(u, v))$ is single-valued and analytic in a suitable neighborhood of the origin. It follows next that the difference of two analytic functions,

$$f^N g_k(u, v) = \varphi(u, v) - F(f),$$

is likewise analytic in this same region. Consequently we may write the transformation T:

$$x = f(u, v), \qquad y = \varphi(u, v) \equiv F(f) + f^N g_k(u, v),$$

whence we have formally

$$J(u, v) \equiv f_u\varphi_v - f_v\varphi_u \equiv f^N(f_u g_{kv} - f_v g_{ku}).$$

This, together with the hypothesis that $J(u, v) \not\equiv 0$, establishes two things: (1) N is not greater than the greatest power of $f(u, v)$ that is contained as a factor in $J(u, v)$, therefore N is finite; (2) $J(f, g_k) \not\equiv 0$ and hence, in particular, $g_k(u, v)$ is not a constant. In the identity (5), $1 \overline{\gtrless} k \overline{\gtrless} N$, hence $g_k(u, v)$ is the result of finite processes of the sort there indicated, and is therefore single-valued and analytic at the origin.

The following factoring for T is now apparent:

$$\begin{aligned}
u_0 &= f(u, v), & v_0 &= g_k(u, v),\\
u_1 &= u_0, & v_1 &= u_0^{n_k}v_0,\\
u_2 &= u_1, & v_2 &= v_1 + C_k u_1^{n_k},\\
&\cdots & &\cdots\\
u_{2i+1} &= u_{2i}, & v_{2i+1} &= (u_{2i})^{n_{k-i}}v_{2i},\\
u_{2i+2} &= u_{2i+1}, & v_{2i+2} &= v_{2i+1} + C_{k-1}(u_{2i+1})^{n_{k-i}},\\
&\cdots & &\cdots\\
u_{2k-1} &= u_{2k-2}, & v_{2k-1} &= (u_{2k-2})^{n_1}v_{2k-2},\\
x &= u_{2k-1}, & y &= v_{2k-1} + C_1(u_{2k-1})^{n_1},
\end{aligned}$$

where, as we have seen in section 2, any transformation

$$u_{2i+1} = u_{2i}, \qquad v_{2i+1} = (u_{2i})^{n_{k-i}}v_{2i}$$

is equivalent to n_{k-i} transformations of the type $x = u$, $y = uv$.

If $g_k(u, v)$ is not a factor of $f(u, v)$ in the point $(0, 0)$, the factoring of T is *complete*. If $g_k(u, v)$ is a factor of $f(u, v)$ in this point we must further factor

a: $\qquad u_0 = f(u, v), \qquad v_0 = g_k(u, v).$

The Jacobian $J(f, g_k)$ can not be identically zero in R, hence all the previous discussion for Case B is valid for the transformation a. Corresponding to a in the factoring of T we shall have here a transformation

a': $\qquad v_0' = g_k(u, v), \qquad u_0' = f_p(u, v),$

where $g_k(u, v)$ is not a factor of $f_p(u, v)$ in the point $(0, 0)$. We must finally, in a finite number of steps, come to a transformation

a'': $\qquad U = F(u, v), \qquad V = G(u, v),$

for which $F(u, v)$ is not a factor of $G(u, v)$ and $G(u, v)$ is not a factor of $F(u, v)$ in the point $(0, 0)$. For suppose the Weierstrassian preparation theorem* gives

$$f(u, v) \equiv [u^r + c_1(v)u^{r-1} + \cdots + c_r(v)]C(u, v),$$
$$g_k(u, v) \equiv [u^s + d_1(v)u^{s-1} + \cdots + d_s(v)]D(u, v),$$

where neither $C(u, v)$ nor $D(u, v)$ is zero in R. If $g_k(u, v)$ is a factor of $f(u, v)$ while $f(u, v)$ is not a factor of $g_k(u, v)$, it follows that s is less than r. Similarly, if $f_p(u, v)$ is a factor of $g_k(u, v)$ and if

$$f_p(u, v) \equiv [u^t + e_1(v)u^{t-1} + \cdots + e_t(v)]E(u, v),$$

then t must be less than s which is less than r. Proceeding in this way, we see that we must in not more than r steps arrive at a transformation for which one of the Weierstrassian polynomials is of degree unity, unless a transformation satisfying the conditions of a'' is earlier obtained. For definiteness, suppose we have

$$u'' = \bar{f}(u, v), \qquad v'' = \bar{g}(u, v),$$

where

$$\bar{f}(u, v) \equiv [u + A(v)]H(u, v), \qquad H(0, 0) \neq 0,$$

and where $\bar{f}(u, v)$ is a factor of $\bar{g}(u, v)$ in the point $(0, 0)$. Applying the initial discussion for Case B (which is again valid since the Jacobian for this transformation can not vanish identically) we secure a transformation

$$u_0'' = \bar{f}(u, v), \qquad v_0'' = \bar{g}_j(u, v),$$

where $\bar{f}(u, v)$ is not a factor of $\bar{g}_j(u, v)$ in the point $(0, 0)$. If $\bar{g}_j(u, v)$ is a factor of $\bar{f}(u, v)$ in this point, it must also be linear in u in the neighborhood of the origin, and $\bar{f}(u, v)$ and $\bar{g}_j(u, v)$ must be equivalent. This is in direct contradiction to the fact that $\bar{f}(u, v)$ is not a factor of $\bar{g}_j(u, v)$ in the point $(0, 0)$. This establishes the existence of a transformation a''. We have therefore proved

THEOREM 1. *The transformation T can always be replaced by a transformation*

$$T': \qquad U = F(u, v), \qquad V = G(u, v),$$

where $F(u, v)$ and $G(u, v)$ are single-valued and analytic in the point $(0, 0)$ *and both vanish there, but where neither is a factor of the other in this point, followed by a finite number m of transformations of the form*

$$x = u, \qquad y = uv$$

* Weierstrass, Abhandlungen aus der Funktionenlehre, p. 105; Goursat, Cours d'Analyse, vol. 2, Sec. 356; Bliss, Bulletin of the American Mathematical Society, vol. 16 (1910), p. 356; Macmillan, ibid., vol. 17 (1910), p. 116.

combined with not more than m transformations of the form

$$x = u, \qquad y = v + cu^n.$$

The transformation T' either is one-to-one and analytic both ways or is again of type T.

4. Case A.

Because of the preceding section, the solution of our problem will be attained when we have a discussion of the nature of the inverse for Case A. In discussing Case A, we note that any transformation T is of one of the following two classes:

(I) At least one element of the determinant J, say f_u, is not zero at the origin.

(II) Every element of the determinant J is zero at the origin.

In considering Case IA we therefore have:

T'': $$x = f(u, v), \qquad y = \varphi(u, v),$$
where
$$f_u(0, 0) \neq 0$$
and where $f(u, v)$ is not a factor of $\varphi(u, v)$ in the point $(0, 0)$.

This is the special case $n = 2$ of the following theorem:* The transformation

$$x_1 = f_1(y_1, \cdots, y_n), \qquad x_2 = f_2(y_1, \cdots, y_n), \qquad \cdots, \qquad x_n = f_n(y_1, \cdots, y_n),$$

where

(1) f_i is a single-valued and analytic function of $y_1, \cdots, y_n$ in the point $(0, \cdots, 0)$ and vanishes there $(i = 1, 2, \cdots, n)$;

$$J_1(y_1, \cdots, y_n) \equiv \frac{D(f_1, \cdots, f_n)}{D(y_1, \cdots, y_n)} = 0 \text{ when } (y) = 0,$$

. . . .

(2) $$J_{m-1}(y_1, \cdots, y_n) \equiv \frac{D(J_{m-2}, f_2, \cdots, f_n)}{D(y_1, \cdots, y_n)} = 0 \text{ when } (y) = 0,$$

$$J_m(y_1, \cdots, y_n) \equiv \frac{D(J_{m-1}, f_2, \cdots, f_n)}{D(y_1, \cdots, y_n)} \neq 0 \text{ when } (y) = 0;$$

has an m-valued continuous inverse, defined throughout a complete neighborhood of $(x) = 0$. This inverse is analytic, with m distinct branches except along a complex $(n - 1)$-dimensional locus, where it is continuous and less than m-valued. Finally, $(y) = 0$ when $(x) = 0$.

* G. R. Clements, Bulletin of the American Mathematical Society, loc. cit., p. 454; Transactions, ibid., vol. XIV (1913), p. 341.

T'' evidently fulfills condition (1) of this theorem; to see that it also satisfies condition (2), make the factoring

$$a: \qquad u_0 = f(u, v), \qquad v_0 = v,$$

$$b: \qquad x = u_0, \qquad y = \Phi(u_0, v_0).$$

The factor a is one-to-one and analytic both ways, with inverse

$$u = h(u_0, v_0), \qquad v = v_0.$$

Since $f_u(0, 0) \neq 0$, $f(u, v)$ is irreducible in R. $\Phi(0, v_0) \not\equiv 0$, for if $\varphi[h(u_0, v_0), v_0]$ is zero wherever $u_0 = f(u, v)$ (irreducible) is zero, then $\varphi(u, v)$ admits $f(u, v)$ as a factor contrary to hypothesis. Therefore $\Phi(0, v_0) \not\equiv 0$ and there is a first coefficient b_m in its series expansion,

$$\Phi(u_0, v_0) = \sum_{n=0}^{\infty} b_n(u_0)v_0^{\,n},$$

which is not zero when $u_0 = 0$. The vanishing of J_i in the point (0, 0) is invariant under the transformation a, therefore

$$J_1(0, 0) = \cdots = J_{m-1}(0, 0) = 0, \qquad J_m(0, 0) \neq 0.$$

Hence for this transformation IA there exists an m and a sequence of functions satisfying condition (2) of the theorem quoted above, and all the requirements of that theorem are fulfilled. Since the inverse can not be single-valued and analytic,* m must be an integer not less than two. This proves

THEOREM 2. *If the transformation T satisfies the further conditions*

$$(1) \qquad f_u(0, 0) \neq 0,$$

$$(2) \qquad f(u, v) \text{ not a factor of } \varphi(u, v) \text{ in the point } (0, 0),$$

then, for the sequence of functions

$$J_1(u, v) = \frac{D(f, \varphi)}{D(u, v)}, \qquad \cdots, \qquad J_{n+1}(u, v) = \frac{D(f, J_n)}{D(u, v)}$$

there exists an integer $m \geqq 2$ such that

$$J_1(0, 0) = \cdots = J_{m-1}(0, 0) = 0, \qquad J_m(0, 0) \neq 0,$$

and for the transformation there exists an m-valued continuous inverse defined throughout a complete neighborhood of $x = 0$, $y = 0$. This inverse is analytic, with m distinct determinations except along a complex one-dimensional locus, where it is continuous and less than m-valued. Finally $u = 0$, $v = 0$ when $x = 0$, $y = 0$.

* Bulletin, loc. cit., p. 452, Theorem 1.

Further factoring for a special class of transformations IA is exhibited in the following theorem.*

THEOREM 3. *If in the transformation of Theorem 2, $J_{m-1}(u, v)$ is a factor in the point $(0, 0)$ of every $J_n(u, v)$ with smaller subscript, then T can be replaced by transformations one-to-one and analytic both ways, combined with one transformation of the form $x = u$, $y = v^m$.*

Replace T by the factors

$$a: \qquad u_0 = f(u, v), \qquad v_0 = v,$$

$$b: \qquad x = u_0, \qquad y = \Phi(u_0, v_0).$$

Set

$$c: \qquad u_0 = u_1, \qquad v_0 = a_0(u_1) + a_1(u_1)v_1,$$

where a_0 and a_1 are functions to be determined. If

$$\Phi(u_0, v_0) = \sum_{n=0}^{\infty} b_n(u_0)v_0^n,$$

formal substitution yields

$$\begin{aligned}\Phi[u_1, a_0(u_1) + a_1(u_1)v_1] &= \sum_{n=0}^{\infty} b_n \cdot (a_0 + a_1 v_1)^n = \sum_{n=0}^{\infty} b_n a_0^n \\ &+ a_1 v_1 \sum_{n=0}^{\infty} n b_n a_0^{n-1} + (a_1^2/2)v_1^2 \sum_{n=0}^{\infty} n(n-1)a_0^{n-2} + \cdots = \Phi(u_1, a_0) \\ &+ a_1 \Phi_{a_0}(u_1, a_0)v_1 + \cdots + (a_1^n/n!)\Phi_{a_0^n}(u_1, a_0)v_1^n + \cdots.\end{aligned} \tag{9}$$

$\Phi_{a_0^{m-1}}(u_1, a_0)$ is single-valued and analytic in the point $u_1 = 0$, $a_0 = 0$ and vanishes there; and further has a derivative with respect to a_0 which is different from zero in this point. Consequently there exists a function $a_0(u_1)$, single-valued and analytic in the point $u_1 = 0$ and vanishing there, for which

$$\Phi_{a_0^{m-1}}[u_1, a_0(u_1)] \equiv 0.$$

Because of the hypothesis on the J_n, $\Phi_{a_0^{m-1}}$ is a factor of every partial derivative of Φ with respect to a_0 of order lower than $m - 1$. Consequently this determination of $a_0(u_1)$ requires that the coefficients of $v_1, \cdots, v_1^{m-2}$ be also identically zero. If, therefore, we assign to $a_1(u_1)$ an arbitrary analytic function of u_1 which does not vanish in R, the transformation c is one-to-one and analytic both ways, transforms origin into origin and transforms $\Phi(u_0, v_0)$ into an analytic function

$$P(u_1, v_1) \equiv B_0(u_1) + B_m(u_1)v_1^m + B_{m+1}(u_1)v_1^{m+1} + \cdots, \tag{10}$$

where

$$B_m(u_1) \equiv \frac{a_1(u_1)^m}{m!} \Phi_{a_0^m}(u_1, a_0(u_1)),$$

* Presented to the American Mathematical Society on October 26, 1912.

is not zero in R. That is, b becomes

$$x = u_1, \qquad y = B_0(u_1) + B_m(u_1)v_1^m + \cdots, \qquad B_m(0) \neq 0.$$

Make the transformation

$$d: \qquad u_2 = u_1, \qquad v_2 = v_1 \sqrt[m]{B_m(u_1) + B_{m+1}(u_1)v_1 + \cdots};$$

$$s: \qquad u_3 = u_2, \qquad v_3 = v_2^m;$$

$$e: \qquad x = u_3, \qquad y = B_0(u_3) + v_3.$$

Then, in symbols,

$$T = a \cdot c^{-1} \cdot d \cdot s \cdot e, \qquad T^{-1} = e^{-1} \cdot s^{-1} \cdot d^{-1} \cdot c \cdot a^{-1}.$$

s is the only one of these transformations which is not one-to-one and analytic both ways. Hence the theorem is proved.

If $m = 2$, the hypotheses of this theorem are satisfied identically.*

We turn next to Case IIA, a part of which is taken care of by the following theorem:† If in the transformation T, $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$, and if R be suitably restricted, then there exists an inverse, defined throughout the (complete) neighborhood of $x = 0$, $y = 0$, everywhere continuous in that neighborhood, finitely multiple-valued but not single-valued, analytic except along a complex one-dimensional locus, and having the value $u = 0$, $v = 0$ when $x = 0$, $y = 0$.

There remain those transformations of Case IIA for which $f(u, v)$ and $\varphi(u, v)$ have a common factor in the point $(0, 0)$. The origin is evidently an explosive point for the inverse; examples indicate that in any other point of $\bar{R}$ there exists a finitely multiple-valued inverse, the order of this multiplicity being in general different in different sub-regions of $\bar{R}$. I hope to give this theory more fully at a later time.

5. Special Transformations.

In section 2 we exhibited at least a partial factoring for any transformation in which either $f(u, v)$ or $\varphi(u, v)$ is the integral power (greater than one) of an analytic function of u and v. The factoring there made

* For this special case $m = 2$, the corresponding theorem for transformations in two and in three real variables has been given respectively by L. S. Dederick, Harvard doctoral thesis (1909), "Certain singularities of transformations of two real variables," p. 124; and by S. E. Urner, "Certain singularities of point transformations in space of three dimensions," Transactions of the American Mathematical Society, Vol. XIII (1912), p. 257, Ex. a. The extension of Mr. Urner's example b of which he says: "There seems to be no theorem in this case analogous to that stated in (a)," is found in my theorem 3. This case $m = 2$ was also discussed for a transformation in n real variables, by Mr. Dederick in the Transactions, Vol. XIV, p. 143.

† G. R. Clements, Transactions, loc. cit., p. 328.

is complete for t if it is complete for

$$u_0 = h(u, v), \qquad v_0 = g(u, v).$$

In sections 3 and 4 we have shown that every transformation of Class I is completely factorable. An immediate corollary of these two facts is

THEOREM 4. *A transformation T which has the form*

$$x = f(u) \equiv a_n u^n + a_{n+1} u^{n+1} + \cdots \; (a_n \neq 0),$$
$$y = \varphi(u, v),$$

can be replaced by two transformations, $T = a \cdot b$, where

$$a: \qquad u_1 = u \sqrt[n]{a_n + a_{n+1} u + \cdots}, \qquad v_1 = \varphi(u, v),$$

$$b: \qquad x = u_1^n, \qquad y = v_1.$$

(a) has an inverse defined by analytic functions, or is completely factorable.

To illustrate the way in which the inverse can be built up for a transformation of type B which has been completely factored, we derive

THEOREM 5. *If the transformation T has the form*

$$x = f(u, v), \qquad y = [f(u, v)]^n g(u, v),$$

where

$$f_u(0, 0) \neq 0, \qquad g(0, 0) = 0$$

and where $f(u, v)$ is not a factor of $g(u, v)$ in the point $(0, 0)$, then to the point $x = 0$, $y = 0$, corresponds the locus $f(u, v) = 0$, and to any other point (x, y)

(1) *if $f_u g_v - f_v g_u \neq 0$ when $u = 0$, $v = 0$, there corresponds one and only one point (u, v), u and v being defined by functions of x and y, single-valued and analytic in this point;*

(2) *if $f_u g_v - f_v g_u = 0$ when $u = 0$, $v = 0$, there correspond in general m points, $m \geqq 2$. The inverse is m-valued and analytic in this point unless it be on a certain complex one-dimensional locus, where it is continuous and less than m-valued.*

$\bar{R}$ is a partial neighborhood of the point $x = 0$, $y = 0$.

A factoring is

$$a: \qquad u_0 = f(u, v), \qquad v_0 = g(u, v),$$

$$b: \qquad x = u_0, \qquad y = u_0^n v_0.$$

In case (1) there exists for a an inverse

$$u = \bar{f}(u_0, v_0), \qquad v = \bar{\varphi}(u_0, v_0),$$

defined by functions single-valued and analytic in the neighborhood of $u_0 = 0$, $v_0 = 0$. For b there exists the inverse $u_0 = x$, $v_0 = y/x^n$, single-valued and analytic in x and y in every point of $\bar{R}$ except the origin (see

section 2). Hence u and v are defined as single-valued and analytic functions of x and y in any point of $\bar{R}$ other than $x = 0$, $y = 0$. This point explodes into the locus

$$u_0 = f(u, v) = 0.$$

In case (2), a satisfies the hypotheses of Theorem 2. Therefore a has a finitely multiple-valued (suppose m-valued) inverse defined throughout the complete neighborhood of $u_0 = 0$, $v_0 = 0$, analytic with m distinct determinations except along a complex one-dimensional locus $D(u_0, v_0) = 0$, where it is continuous and less than m-valued. Further u_0 and v_0 are single-valued and analytic in x and y at every point (x, y) of $\bar{R}$ except the origin. Consequently the inverse defines u and v as in general m-valued and analytic in x and y in a point of $\bar{R}$; in points of a complex one-dimensional locus, excluding the origin, the inverse ceases to be analytic but is continuous and less than m-valued; the point $x = 0$, $y = 0$, explodes into the locus $f(u, v) = 0$.

Because of the nature of b, $\bar{R}$ is in either case the partial neighborhood of $x = 0$, $y = 0$.

We give the detailed discussion for a special case of type IIA.

THEOREM 6. *If the transformation T has the form*

$$x = f(u, v) \equiv c_{20}u^2 + c_{11}uv + c_{02}v^2 + c_{30}u^3 + \cdots,$$

$$y = \varphi(u, v) \equiv d_{20}u^2 + d_{11}uv + d_{02}v^2 + d_{30}u^3 + \cdots,$$

where the terms quadratic in u and v are not identically zero for either $f(u, v)$ or $\varphi(u, v)$ and where these quadratic terms have no common factor in the point $(0, 0)$, *then there exists a four-valued continuous inverse, defined throughout the complete neighborhood of $x = 0$, $y = 0$. This inverse is analytic, with four distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than four-valued. Finally, $u = 0$, $v = 0$ when $x = 0$, $y = 0$.*

Because of the hypotheses on the quadratic forms

$$f_2 \equiv c_{20}u^2 + c_{11}uv + c_{02}v^2$$

and

$$\varphi_2 \equiv d_{20}u^2 + d_{11}uv + d_{02}v^2,$$

it is always possible to secure from T by a linear transformation in x and y whose Jacobian is not zero, a second transformation T'' for which one of the quadratic forms involved is a perfect square while the second quadratic form is not a perfect square and does not admit the square root of the first form as a factor. Then by a linear transformation in u and v whose Jacobian

is not zero, this can be brought to the form

$$T': \qquad x = f(u', v') \equiv v'^2 + c_{30}u'^3 + c_{21}u'^2v' + \cdots,$$
$$y = \varphi(u', v') \equiv d_{20}u'^2 + d_{11}u'v' + d_{02}v'^2 + d_{30}u'^3 + \cdots,$$

where $d_{20} \neq 0$, $d_{11} \neq 0$, $d_{02} \neq 0$. Of course f, φ, the c's and d's have a different definition in T' than they have in T of the theorem. The Jacobian for T' is

$$J = -4d_{20}u'v' - 2d_{11}v'^2 + \cdots. \tag{11}$$

Seek for the transformation

$$u' = u + sv, \qquad v' = r + v, \tag{12}$$

so to determine r and s that the coefficients of the first power of v in the series into which (12) carries the right members of T', shall be identically zero. If we write T' as

$$\begin{aligned} x &= f(u', v') \equiv a(u') + a'(u')v' + a''(u')v'^2 + \cdots, \\ y &= \varphi(u', v') \equiv b(u') + b'(u')v' + b''(u')v'^2 + \cdots, \end{aligned} \tag{13}$$

formal substitution of (12) in $f(u', v')$ yields

$$x = f(u', v') = f(u + sv, r + v) = \sum_{n=0}^{\infty} a^{(n)}(u + sv)(r + v)^n,$$

whence the coefficient of v is

$$\begin{aligned} &a_u(u)s + [a_u'(u)sr + a'(u)] + [a_u''(u)sr^2 + a''(u)2r] \\ &\qquad + [a_u'''(u)sr^3 + a'''(u)3r^2] + \cdots \\ &= s(a_u + a_u'r + a_u''r^2 + a_u'''r^3 + \cdots) + (a' + 2a''r + 3a'''r^2 + \cdots) \\ &= sf_u(u, r) + f_r(u, r). \end{aligned} \tag{14}$$

Similarly, for φ the coefficient of v is

$$s\varphi_u(u, r) + \varphi_r(u, r). \tag{15}$$

Setting (14) and (15) each equal to zero and eliminating s, we have as a necessary condition for the determination of r

$$f_u(u, r) \cdot \varphi_r(u, r) - f_r(u, r) \cdot \varphi_u(u, r) \equiv J(u, r) = 0. \tag{16}$$

But

$$J(u, r) = -4d_{20}ur - 2d_{11}r^2 + \cdots.$$

The solutions r of $J(u, r) = 0$, which are zero when $u = 0$, are had from*

$$\frac{J(u, r)}{-2d_{11}} \equiv [r^2 + A(u)r + B(u)]e^{w(u, r)}, \qquad w(0, 0) = 0. \tag{17}$$

* Weierstrass's preparation theorem.

Comparison of the two members of this identity shows that

$$A(0) = 0, \quad A_u(0) = 2d_{20}/d_{11} \neq 0, \quad B(0) = 0, \quad B_u(0) = 0, \quad B_{u^2}(0) = 0.$$

That is, $A(u)$ and $B(u)$ have the form

$$A(u) \equiv (2d_{20}/d_{11})u + hu^2 + h'u^3 + \cdots,$$

$$B(u) \equiv mu^3 + m'u^4 + \cdots.$$

Solving

$$r^2 + A(u)r + B(u) = 0$$

for r, we obtain

$$(18) \quad \begin{aligned} r &= \tfrac{1}{2}[-A(u) \pm \sqrt{A^2(u) - 4B(u)}] \\ &= \tfrac{1}{2}[-(2d_{20}/d_{11})u - hu^2 - \cdots \pm (2d_{20}/d_{11})u\sqrt{1 + nu + n'u^2 + \cdots}]. \end{aligned}$$

If we take the lower sign with the radical

$$r' = -(2d_{20}/d_{11})u + pu^2 + p'u^3 + \cdots;$$

if we take the upper sign

$$(19) \qquad r'' = qu^2 + q'u^3 + \cdots.$$

To determmine s consider in more detail the equation

$$s\varphi_u(u, r) + \varphi_r(u, r) = 0,$$

where r is replaced by $r''(u)$.*

$$\varphi_u(u, r) \equiv b_u + b_u'r + b_u''r^2 + \cdots = u(2d_{20} + g'u + g''u^2 + \cdots),$$

$$\varphi_r(u, r) \equiv b' + 2b''r + 3b'''r^2 + \cdots = u(d_{11} + h'u + h''u^2 + \cdots).$$

Therefore

$$s\varphi_u(u, r) + \varphi_r(u, r) = u[s(2d_{20} + g'u + g''u^2 + \cdots) + (d_{11} + h'u + \cdots)].$$

Since $u \not\equiv 0$, this will be identically zero throughout R if and only if

$$(20) \quad s = \frac{-(d_{11} + h'u + h''u^2 + \cdots)}{2d_{20} + g'u + g''u^2 + \cdots} = -\frac{d_{11}}{2d_{20}} + k'u + k''u^2 + \cdots.$$

This pair of values for r and s also makes $sf_u(u, r) + f_r(u, r)$ identically zero. For

$$J(u, r''(u)) \equiv 0,$$

that is, when r is replaced by $r''(u)$,

$$f_u\varphi_r - f_r\varphi_u \equiv 0.$$

* The use of $r''(u)$ rather than $r'(u)$ is not accidental. If $r'(u)$ be used, s is not defined when $u = 0$.

Since each derivative in the left member of this identity has u as a factor, let

$$f_u(u, r''(u)) \equiv u\bar{f}_u, \qquad f_r(u, r''(u)) \equiv u\bar{f}_r,$$

$$\varphi_u(u, r''(u)) \equiv u\bar{\varphi}_u, \qquad \varphi_r(u, r''(u)) \equiv u\bar{\varphi}_r,$$

where

$$\bar{\varphi}_u(0) = 2d_{20} \neq 0.$$

Then

$$J(u, r''(u)) \equiv u^2(\bar{f}_u\bar{\varphi}_r - \bar{f}_r\bar{\varphi}_u) \equiv 0.$$

Since $u \not\equiv 0$, it follows that

$$\bar{f}_u\bar{\varphi}_r - \bar{f}_r\bar{\varphi}_u \equiv 0.$$

But

$$s = -(\bar{\varphi}_r/\bar{\varphi}_u).$$

Therefore, when $r = r''(u)$,

$$sf_u + f_r \equiv u(s\bar{f}_u + \bar{f}_r) \equiv u\left(\frac{-\bar{\varphi}_r\bar{f}_u}{\bar{\varphi}_u} + \bar{f}_r\right) \equiv (-u/\bar{\varphi}_u)(\bar{f}_u\bar{\varphi}_r - \bar{f}_r\bar{\varphi}_u) \equiv 0.$$

Therefore the transformation (12) is completely determined. Its Jacobian $1 - r_u s + s_u v$ has the value 1 when $u = 0$, $v = 0$. Consequently (12) is one-to-one and analytic both ways. Since it further carries origin into origin, the results of this substitution in $f(u', v')$ and $\varphi(u', v')$ will each be analytic and take the value zero when $u = 0$, $v = 0$. The terms involving u alone are

$$f(u, r''(u)) \equiv c_{30}u^3 + Pu^4 + \cdots, \qquad \varphi(u, r''(u)) \equiv d_{20}u^2 + Qu^3 + \cdots, \qquad d_{20} \neq 0.$$

The coefficient of v^2 is computed to be

$$\frac{s^2}{2}a_u + \left[\frac{s^2}{2}ra_{u^2}' + sa_u'\right] + \left[\frac{s^2r^2}{2}a_{u^2}'' + 2rsa_u'' + a''\right]$$

$$+ \left[\frac{s^2}{2}r^3a_{u^2}''' + 3sr^2a_u''' + 3ra'''\right] + \cdots$$

$$(21) \qquad = \frac{s^2}{2}[a_{u^2} + a_{u^2}'r + a_{u^2}''r^2 + a_{u^2}'''r^3 + \cdots]$$

$$+ s[a_u' + 2a_u''r + 3a_u'''r^2 + \cdots] + [a'' + 3a'''r + \cdots]$$

$$= \frac{s^2}{2}f_{u^2}(u, r) + sf_{ur}(u, r) + \tfrac{1}{2}f_{r^2}(u, r),$$

with a corresponding expression for the coefficient of v^2 in φ. When we replace r and s by their computed values from (19) and (20) and set $u = 0$, this coefficient for v^2 becomes

for the transform of $f \cdots \frac{1}{2} \cdot 2 = 1$,

for the transform of $\varphi \cdots \frac{1}{2} \cdot 2d_{02} \neq 0$.

We have then brought T' to the form

$$(22)\qquad \begin{aligned} x &= (c_{30}u^3 + Pu^4 + \cdots) + v^2(1 + e_{10}u + e_{01}v + \cdots), \\ y &= (d_{20}u^2 + Qu^3 + \cdots) + v^2(d_{02} + \cdots). \end{aligned}$$

Set

$$(23)\qquad u_1 = u\sqrt{d_{20} + Qu + \cdots}, \qquad v_1 = v\sqrt{d_{02} + \cdots},$$

a transformation one-to-one and analytic both ways. Then (22) becomes

$$(24)\qquad x = F(u_1) + v_1^2 G(u_1, v_1), \qquad y = u_1^2 + v_1^2,$$

where $G(0, 0) \neq 0$, and $F(u_1)$ has u_1^3 as a factor. (For the second equation of (23) makes it evident that $v = 0$ when $v_1 = 0$, $u_1 = u$. Since further the Jacobian for the inverse of (23) can not vanish in the origin, it follows that this inverse must have the form

$$u = g(u_1), \qquad v = v_1 h(u_1, v_1), \qquad h(0, 0) \neq 0.$$

The conclusions concerning (24) are then immediate.) To find the inverse of (24) we have to solve for u and v the equations

$$(25)\quad F(u) + v^2 G(u, v) - x = 0, \qquad u^2 + v^2 - y = 0, \qquad G(0, 0) \neq 0,$$

where we have dropped the subscripts on u and v. The solutions v of the first equation, which are zero when $u = 0$, $x = 0$, using the Weierstrassian preparation theorem, are had from

$$(26)\qquad v^2 + C(u, x)\, v + D(u, x) = 0,$$

where

$$(27)\qquad F(u) + v^2 G(u, v) - x \equiv [v^2 + C(u, x)\, v + D(u, x)]e^{w(u, v, x)}.$$

The left number of (27) expressed as a power series in u, v, x contains no term involving v to the first power only. Consequently the identity (27) can hold true only if $C(u, x) \equiv 0$. Therefore solutions of (25) are had from

$$(28)\qquad v^2 + D(u, x) = 0, \qquad v^2 + u^2 - y = 0.$$

Eliminating v, we have

$$E(u, x, y) \equiv u^2 - D(u, x) - y = 0.$$

From (27) we find that

$$D(0, 0) = 0, \qquad D_u(0, 0) = 0, \qquad D_{u^2}(0, 0) = 0,$$

whence it follows at once that

$$E(0, 0, 0) = 0, \qquad E_u(0, 0, 0) = 0, \qquad E_{u^2}(0, 0, 0) = 2, \qquad E(u, 0, 0) \not\equiv 0.$$

Since

$$E_y(0, 0, 0) = -1,$$

$E(u, x, y)$ is irreducible. Consequently the discriminant $\Delta(x, y)$ of $E(u, x, y)$, which can be obtained by eliminating u from $E(u, x, y)$ and $E_u(u, x, y)$, is not identically zero. But $\Delta(0, 0) = 0$, therefore $\Delta(x, y) = 0$ defines a locus passing through the origin in the xy-space. $E(u, x, y) = 0$ defines u as a two-valued function of x and y, continuous throughout a complete neighborhood of $x = 0$, $y = 0$, analytic with two distinct determinations except for points of $\Delta(x, y) = 0$, where the two determinations of the root become coincident. Since

$$E(u(x, y), x, y) \equiv 0,$$

i. e., since

$$[u(x, y)]^2 - y \equiv D(u(x, y), x),$$

the form of (28) shows that the determination of v from either equation of (28) will satisfy the other equation also. v, the square root of a function continuous throughout the complete neighborhood of the origin, is itself continuous throughout the same region. It is four-valued in any point in which $[u(x, y)]^2 - y$ is two-valued, two-valued in any point in which $[u(x, y)]^2 - y$ is single-valued and different from zero, and has the value zero when

$$[u(x, y)]^2 - y = 0.$$

$x = 0$, $y = 0$ is a point of this last locus. In the neighborhood of any point of this region not on

$$\Delta(x, y) = 0,$$

the equations (28) have four distinct analytic solutions, which we may indicate schematically as

$$(u_0, v_0), \quad (u_0, v_1), \quad (u_1, v_2), \quad (u_1, v_3)$$

or

$$(u_0, v_0), \quad (u_0, v_1), \quad (u_1, v_0), \quad (u_1, v_1).$$

Hence the theorem is proved.

If instead of the transformation T we consider the transformation T_0 whose definition differs from that for T in that it omits the condition (c), we can deduce a theorem of which Theorem 6 is a special case.

Theorem 7. *If the transformation T_0 has the form*

$$x = f(u, v) \equiv f_m(u, v) + f_{m+1}(u, v) + \cdots,$$
$$y = \varphi(u, v) \equiv \varphi_n(u, v) + \varphi_{n+1}(u, v) + \cdots,$$

where $f_i(u, v)$ and $\varphi_i(u, v)$ are homogeneous polynomials of degree i in u and v $(i = m, m + 1, \cdots; i = n, n + 1, \cdots)$, and where $f_m(u, v)$ and $\varphi_n(u, v)$ have no common factor, then there exists an mn-valued inverse, defined through-

out the complete neighborhood of $x = 0$, $y = 0$. *This inverse is analytic with* mn *distinct branches except (if* $mn > 1$) *along a complex one-dimensional locus where it is continuous and less than* mn*-valued. Finally* $u = 0$, $v = 0$ *when* $x = 0$, $y = 0$.

Since $f_m(u, v)$ and $\varphi_n(u, v)$ have no common factor, it follows a fortiori that $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$. But I have shown in the theorem quoted near the end of section 4 that if $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$ and if R be suitably restricted, then there exists an inverse defined throughout the complete neighborhood of $x = 0$, $y = 0$, everywhere continuous in that neighborhood, finitely multiple-valued, analytic except (if more than one-valued) along a complex one-dimensional locus where roots elsewhere distinct become coincident, and having the value $u = 0$, $v = 0$ when $x = 0$, $y = 0$. The proof of this is independent of any assumption concerning the value of $J(u, v)$ in R.*

$J(u, v)$ can not be identically zero; for if it were then y would be expressible† as an analytic function of x alone in a suitable neighborhood of $x = x_1 = f(u_1, v_1)$, where (u_1, v_1) is a point of R in which a first derivative of $f(u, v)$ or of $\varphi(u, v)$, say $f_u(u, v)$, is not zero. The same point (x, y) would then correspond to every point (u, v) which is a point of the locus $f(u, v) = x_1$ and is within a suitable neighborhood S of (u_1, v_1) in which $f_u(u, v) \neq 0$. This contradicts the existence of a finitely multiple-valued inverse, hence $J(u, v) \not\equiv 0$.

If $J(0, 0) \neq 0$, then $m = n = 1$ and this inverse is single-valued and analytic throughout the region $\bar{R}$ in which it is defined; if $J(0, 0) = 0$, it is multiple valued and satisfies the properties required by the theorem except that as yet we have no count of the *number* of roots included in this inverse. For the case under discussion, in which $f_m(u, v)$ and $\varphi_n(u, v)$ have no common factor, this number is known‡ to be exactly mn. Further, in any point of the region of definition in which $J(u, v) \neq 0$, these roots are distinct. Hence the theorem is proved.

UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN.

*Transactions, loc. cit., p. 333.

†Jordan, Cours d'Analyse, vol. 1, 2d ed. (1893), p. 86.

‡G. A. Bliss, Transactions of the American Mathematical Society, Vol. XIII (1912), p. 134.

ON THE PROJECTIVE DIFFERENTIAL GEOMETRY OF PLANE ANHARMONIC CURVES.

By Samuel W. Reaves.

1. Introduction.—Following Halphen and Wilczynski* we shall interpret the fundamental solutions y_1, y_2, y_3 of a linear homogeneous differential equation of the third order

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0 \tag{1}$$

as the homogeneous coördinates of a point P_y. This point will describe an integral curve of (1) when the independent variable passes through all of its values. Associated with the point P_y are two other semi-covariant points† P_z, P_ρ, whose coördinates are given by the expressions

$$\begin{aligned} z_i &= y_i' + p_1y_i, \\ \rho_i &= y_i'' + 2p_1y_i' + p_2y_i \qquad (i = 1, 2, 3), \end{aligned} \tag{2}$$

and which, together with P_y, determine a triangle well suited to serve as triangle of reference for the purpose of investigating the properties of the curve in the vicinity of the point P_y. Of course as the independent variable changes these three points will trace three curves C_y, C_z, C_ρ.

Referred to the triangle $P_yP_zP_\rho$ the equation of the conic which osculates C_y at P_y is‡

$$x_2^2 - 2x_1x_3 + P_2x_3^2 = 0. \tag{3}$$

The covariant $\Theta_3'y + 3\Theta_3z$§ determines a point P_γ whose coördinates are $(\Theta_3', 3\Theta_3, 0)$. It is the point in which the tangent at P_y meets the line upon which lie the three points of inflection of the eight-pointic nodal cubic of P_y.

The coördinates of the Halphen point P_h of P_y are||

$$\begin{aligned} x_1 &= 7(5\Theta_3'\Theta_8 - 756\Theta_3^4)^2 + 25\Theta_8^3 + 1575\Theta_3^2\Theta_8^2P_2, \\ x_2 &= 210\Theta_3\Theta_8(5\Theta_3'\Theta_8 - 756\Theta_3^4), \\ x_3 &= 3150\,\Theta_3^2\Theta_8^2. \end{aligned} \tag{4}$$

* Projective Differential Geometry of Curves and Ruled Surfaces, B. G. Teubner, Leipzig, 1906, p. 60.

† Wilczynski, loc. cit., p. 61.

‡ Wilczynski, loc. cit., p. 65.

§ Wilczynski, loc. cit., pp. 85–86.

|| Wilczynski, loc. cit., p. 68.

It is the purpose of this paper to find the equations of the loci C_z, C_ρ, C_γ of P_z, P_ρ, P_γ, the locus $C_{o'}$ of the center O' of the osculating conic, and the locus C_h of the Halphen point P_h, for each of the following anharmonic curves* (also known as W-curves):

$$y = x^\lambda \qquad (\lambda \neq 0, 1, -1, \tfrac{1}{2}, 2), \tag{5}$$

$$r = e^{m\theta}, \tag{6}$$

$$y = e^x. \tag{7}$$

For the logarithmic spirals (6), the osculating conic at an arbitrary point P_y will be studied in some detail and a construction for its center and axes will be given.

Let the sides of our triangle of reference be the line at infinity, the x-axis, and the y-axis, and let us choose the unit point in such a way that $(y_2/y_1, y_3/y_1)$ will be the cartesian coördinates of P_y. The relation between this system of coördinates and the local system determined by the semi-covariants y, z, ρ of the point P_y, is given by the equations

$$\sigma_i = y_i x_1 + z_i x_2 + \rho_i x_3 \qquad (i = 1, 2, 3). \tag{8}$$

2. The Curves $y = x^\lambda$.—Since $x = y_2/y_1$ and $y = y_3/y_1$, this family of curves may be represented parametrically by the equations

$$y_1 = 1, \qquad y_2 = t, \qquad y_3 = t^\lambda, \tag{9}$$

and since y_1, y_2, y_3 are to be solutions of a differential equation of the form (1), we find, on substitution, the following values for the coefficients:

$$p_1 = \frac{2-\lambda}{3t}, \qquad p_2 = p_3 = 0. \tag{10}$$

Making use of equations (2) we find the coördinates of P_z, P_ρ, P_γ to be

$$z_1 = \frac{2-\lambda}{3t}, \qquad z_2 = \frac{5-\lambda}{3}, \qquad z_3 = \frac{2\lambda+2}{3}t^{\lambda-1}; \tag{11}$$

$$\rho_1 = 0, \qquad \rho_2 = \frac{4-2\lambda}{3t}, \qquad \rho_3 = \frac{\lambda^2+\lambda}{3}t^{\lambda-2}; \tag{12}$$

$$\gamma_1 = \lambda + 1, \qquad \gamma_2 = (\lambda-2)t, \qquad \gamma_3 = (1-2\lambda)t^\lambda. \tag{13}$$

Placing in (11) $z_2/z_1 = x$, $z_3/z_1 = y$, and eliminating t, we have for the equation of C_z

$$C_z\colon \quad y = \frac{2\lambda+2}{2-\lambda}\left(\frac{2-\lambda}{5-\lambda}\right)^\lambda x^\lambda. \tag{14}$$

* Lie-Scheffers, Continuierliche Gruppen, pp. 68–82. Enc. der Math. Wiss., III, $D4$, pp. 204–215. Wilczynski, loc. cit., pp. 86–90.

In like manner we have from (12) and (13)

$$(15) \qquad C_\rho: \quad \text{The line at infinity, and}$$

$$(16) \qquad C_\gamma: \quad y = \frac{1-2\lambda}{1+\lambda}\left(\frac{\lambda+1}{\lambda-2}\right)^\lambda x^\lambda.$$

The equation of the osculating conic, as derived from (3) by means of the transformation (8), is

$$(17) \qquad \begin{aligned} &\lambda^2(2\lambda-1)(\lambda+1)t^{2\lambda}x^2 + 4\lambda(\lambda-2)(2\lambda-1)t^{\lambda+1}xy + (\lambda+1)(2-\lambda)t^2y^2 \\ &+ 4\lambda(\lambda^2-1)(2-\lambda)t^{2\lambda+1}x + 4(\lambda^2-1)(1-2\lambda)t^{\lambda+2}y \\ &+ (\lambda-1)^2(\lambda-2)(2\lambda-1)t^{2\lambda+2} = 0. \end{aligned}$$

This conic is an hyperbola, parabola, or ellipse according as the expression $\lambda^2(\lambda-1)^2(\lambda-2)(2\lambda-1)t^{2\lambda+2}$ is positive, zero, or negative. The coördinates of the center O' are

$$(18) \qquad x = \frac{2\lambda+2}{2\lambda-1}t, \qquad y = \frac{2+2\lambda}{2-\lambda}t^\lambda,$$

and the locus $C_{o'}$ of the center has the equation

$$(19) \qquad C_{o'}: \quad y = \frac{2+2\lambda}{2-\lambda}\left(\frac{2\lambda-1}{2\lambda+2}\right)^\lambda x^\lambda.$$

Transforming the coördinates (4) of the Halphen point by means of (8), placing $\sigma_2/\sigma_1 = x$, $\sigma_3/\sigma_1 = y$, and eliminating t, we have for the equation of the locus C_h of P_h

$$(20) \qquad C_h: \quad y = \frac{A}{B}\left(\frac{B}{C}\right)^\lambda x^\lambda,$$

where

$$A = 27{,}876(\lambda^6+1) - 108{,}198(\lambda^5+\lambda) - 69{,}372(\lambda^4+\lambda^2) + 461{,}454\lambda^3,$$

$$B = 27{,}876\lambda^6 - 59{,}058\lambda^5 - 192{,}222\lambda^4 + 340{,}494\lambda^3 + 234{,}918\lambda^2 - 486{,}198\lambda + 162{,}066,$$

$$C = 162{,}066\lambda^6 - 486{,}198\lambda^5 + 234{,}918\lambda^4 + 340{,}494\lambda^3 - 192{,}222\lambda^2 - 59{,}058\lambda + 27{,}876.$$

It is thus seen that *the loci C_γ, $C_{o'}$ and C_h are all anharmonic curves with the same invariant and belonging to the same fundamental triangle as the original curve.*

3. The Logarithmic Spirals: $r = e^{m\theta}$.—The number m, which is the cotangent of the constant angle α which the tangent makes with the radius

vector to the point of contact, may be called the *pitch* of the spiral. By putting

$$\frac{y_2}{y_1} = x = r \cos \theta$$

and

$$\frac{y_3}{y_1} = y = r \sin \theta$$

we may represent this family of spirals parametrically by the equations

$$(21) \qquad y_1 = 1, \qquad y_2 = e^{m\theta} \cos \theta, \qquad y_3 = e^{m\theta} \sin \theta.$$

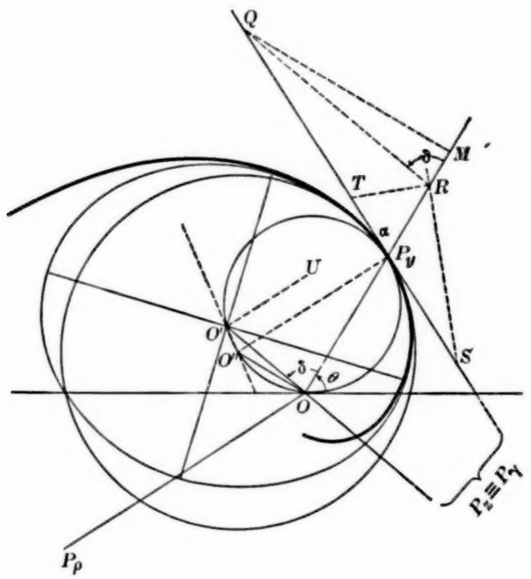
By methods similar to those of last paragraph we find values of $p_i, z_i, \rho_i, \gamma_i$ $(i = 1, 2, 3)$, and by eliminating θ we find the equations of the loci C_z, C_ρ, C_γ. It turns out that these are themselves logarithmic spirals with the same pole and pitch as the original spiral C_y, their equations being

$$(22) \qquad C_z \equiv C_\gamma: \qquad r = \frac{\sqrt{m^2 + 9}}{2m} e^{m(\theta - \beta)},$$

$$(23) \qquad C_\rho: \qquad r = \frac{2}{\sqrt{m^2 + 1}} e^{m\left(\theta - \alpha - \frac{\pi}{2}\right)},$$

where

$$\beta = \cot^{-1} \frac{m}{3}.$$



β and $(\pi/2 + \alpha)$ are the angles which OP_γ and OP_ρ make with OP_y, and these angles are therefore constant.

The equation of the osculating conic at $P_y\ (r, \theta)$ is

$$
\begin{aligned}
&[(17m^2+9)\cos^2\theta + (10m^3-6m)\sin\theta\cos\theta + (2m^4+3m^2+9)\sin^2\theta]x^2 \\
&+2[(3m-5m^3)\cos^2\theta+(14m^2-2m^4)\sin\theta\cos\theta+(5m^3-3m)\sin^2\theta]xy \\
&+[(2m^4+3m^2+9)\cos^2\theta+(6m-10m^3)\sin\theta\cos\theta+(17m^2+9)\sin^2\theta]y^2 \\
&+8[-2m^2\cos\theta+(3m+m^3)\sin\theta]e^{m\theta}x \\
&-8[(3m+m^3)\cos\theta+2m^2\sin\theta]e^{m\theta}y-(m^2+9)e^{2m\theta}=0.
\end{aligned}
\tag{24}
$$

The discriminant of the second degree terms, $ab - h^2$, has the value $9(m^2+1)^2(m^2+9)$, and since this is always positive *the osculating conic at all points of a logarithmic spiral is an ellipse.*

The coördinates of the center O' of this ellipse are

$$
x = \frac{4me^{m\theta}}{m^2+9}(m\cos\theta - 3\sin\theta), \qquad y = \frac{4me^{m\theta}}{m^2+9}(m\sin\theta + 3\cos\theta), \tag{25}
$$

and the locus $C_{o'}$ of the center has the polar equation

$$
C_{o'}: \qquad r = \frac{4m}{\sqrt{m^2+9}}e^{m(\theta-\delta)} \tag{26}
$$

where $\delta = \cot^{-1} m/3$ is the constant angle which OO' makes with OP_y. It may be shown that $\delta = \beta - \pi$.

The angle θ' which either axis of the osculating ellipse makes with the x-axis satisfies the equation

$$
\tan(2\theta' - 2\theta) = \frac{5m^2-3}{m^3-7m}. \tag{27}
$$

Since $(\theta' - \theta)$ is constant, the axes of the ellipse make constant angles with the radius vector OP_y.

The lengths of the semi-axes of the osculating ellipse are found to be

$$
\begin{aligned}
a' &= \sqrt{2}r\sin\delta\,\frac{\sin\frac{\delta}{2}}{\sin\alpha} \\
b' &= \sqrt{2}r\sin\delta\,\frac{\cos\frac{\delta}{2}}{\sin\alpha}
\end{aligned}
\tag{28}
$$

and both axes are therefore proportional to the distance of the point of osculation from the pole. *Hence the osculating ellipse at all points of the logarithmic spiral remains similar to itself,* the constant eccentricity being $\sqrt{1 - \tan^2\delta/2}$.

The center O'' of the osculating circle at P_y is found to have the coördinates

(29) $$x = -me^{m\theta}\sin\theta, \qquad y = me^{m\theta}\cos\theta,$$

and therefore OO'' is perpendicular to OP_y. Hence *the pole O lies on the circle having the radius of curvature $O''P_y$ for diameter*, and it may be shown that *the center O' of the osculating ellipse lies on the same circle.*

To construct the center O' of the osculating ellipse we may proceed as follows: From any point Q on the tangent at P_y drop a perpendicular to OP_y meeting it in M. Divide P_yM at R in the ratio $2:1$. Draw through O a line parallel to RQ to meet the circle $OO''P_y$ again in the required point O'.

The directions of the axes may be found by the following construction: Join $O'O''$ and draw $O'U$ parallel to $O''P_y$. The axes of the osculating ellipse will lie on the internal and external bisectors of the angle $O''O'U$. If $m > 0$, the major axis is the one which cuts OP_y on the side of O towards P_y.

Finally we may find the lengths of the axes geometrically as follows: If the internal and external bisectors of the angle QRM meet P_yQ in S and T, two sides of the triangle RP_yS have the ratio $\sin\delta/2 : \sin\alpha$, and two sides of the triangle RP_yT have the ratio $\cos\delta/2 : \sin\alpha$. If a perpendicular to OO' be dropped from P_y, its length is $r\sin\delta$. Hence, from (28), $a'/\sqrt{2}$ and $b'/\sqrt{2}$ may be found in magnitude by constructing fourth proportionals to these line segments, and then a' and b' may be readily found.

From the fact that the osculating ellipse has constant eccentricity, that its axes make constant angles with the radius vector OP_y, and that its center generates a logarithmic spiral having the same pole and pitch as the original spiral, it may be inferred that a great many other points of the figure will likewise generate spirals of the same pole and pitch. For example, this is true of the foci and extremities of the axes of the ellipse; also for the intersection points of the axes with OP_y, OP_ρ, OO'', $O''P_y$, QP_y, etc. And, dually, there are a great many lines of the figure which envelop spirals of the same pole and pitch as the original one. This is true, for example, of the axes of the ellipse, the lines joining the foci to the extremities of the minor axis, etc.

If we speak of all points and lines thus associated with a particular point P_y of the spiral, and which generate spirals while P_y traces the spiral C_y, as constituting a *set* of generators, we may say more generally, (a) any line through a point generator of a set, making a constant angle with OP_y, is a line generator of the same set; (b) a point dividing in a constant ratio the line segment determined by two point generators of the same set, is a point generator of the set; (c) the point of intersection of two line generators

of the same set is a point generator of the set; and (*d*) the line joining two point generators of the same set is a line generator of the set.

The Halphen point and its locus may be found by the method of the last paragraph. We find

$$(30)\qquad C_h: \quad r = \frac{\sqrt{Q^2+R^2}}{P} e^{m\left(\theta - \tan^{-1}\frac{R}{Q}\right)},$$

where

$$P = 54{,}022m^6 + 497{,}106m^4 + 1{,}147{,}122m^2 + 109{,}350,$$

$$Q = 9{,}292m^6 + 313{,}776\,m^4 + 2{,}184{,}732m^2 - 145{,}800,$$

$$R = -16{,}380m^5 - 325{,}080m^3 + 1{,}122{,}660m,$$

and $\tan^{-1} R/Q$ is the constant angle which OP_h makes with OP_y.

We have now seen that *the loci* C_γ, $C_{o'}$, $C_{o''}$, C_h *are all spirals of the same pole and pitch as the spiral from which they are derived.*

4. The Exponential Curves: $y = e^x$.—We may use the parametric equations

$$(31)\qquad y_1 = 1, \qquad y_2 = t, \qquad y_3 = e^t,$$

and by the methods of paragraph 2 we shall find

$$(32)\qquad C_z \equiv C_\gamma: \quad y = -2e^3e^x$$

C_ρ: The line at infinity.

The equation of the osculating conic is

$$(33)\qquad 2e^{2t}x^2 + 8e^txy + 7y^2 + (10-4t)e^{2t}x + (8-8t)e^ty + (2t^2 - 10t + 17) = 0,$$

and *it is therefore always an hyperbola.*

The locus of the center of the osculating conic and the locus of the Halphen point have respectively the equations

$$(34)\qquad C_{o'}: \quad y = -6e^{x-\frac{29}{2}},$$

$$(35)\qquad C_h: \quad y = \frac{27511}{4646} e^{x+\frac{4095}{2323}}.$$

Thus *the loci* C_γ, $C_{o'}$, C_h *are all anharmonic curves of the same kind as the original curve* C_y.

University of Oklahoma,
September 14, 1912.

ON THE RANK OF A SYMMETRICAL MATRIX.*

BY L. E. DICKSON.

1. A matrix (a_{ij}) is said to be of rank r if some minor M_r of order r is not zero, while every minor M_{r+1} of order $r+1$ is zero. A minor of order r in which there occur exactly k elements a_{ii} of the main diagonal of a square matrix (a_{ij}) shall be designated by $M_r^{(k)}$. In particular, any $M_r^{(r)}$ is called a principal minor.

Of various theorems† on the rank of a symmetrical matrix, the following theorem ‡ due to Kronecker is especially useful:

In a symmetrical matrix of rank $r (r > 0)$, at least one principal minor of order r is not zero.

The following proof rests upon the fact that if n linear homogeneous equations in n variables have a set of solutions not all zero, the determinant of the coefficients is zero.

2. THEOREM. *If, in a symmetrical matrix (a_{ij}), every $M_{r+1} = 0$ and every $M_r^{(r)} = 0$, then every $M_r = 0$.*

To proceed by induction, let ρ be a fixed integer, $0 \leqq \rho < r$, for which every $M_r^{(k)} = 0$, $k > \rho$. We shall prove that every $M_r^{(\rho)} = 0$. The assumption that there is a non-vanishing $M_r^{(\rho)}$ will be shown to involve a contradiction. After a rearrangement of the rows of (a_{ij}) and the like rearrangement of the corresponding columns, a change not affecting our hypotheses, we may assume that

$$M = |a_{i1} \cdots a_{i\rho} a_{ir+1} \cdots a_{i2r-\rho}| \neq 0 \qquad (i = 1, \cdots, r).$$

Let t be one of the integers $r+1, \cdots, 2r-\rho$. Then M is a minor of

$$M_{r+1} = |a_{i1} \cdots a_{i\rho} a_{i\rho+1} a_{ir+1} \cdots a_{i2r-\rho}| = 0 \quad (i = 1, \cdots, r, t).$$

Expand the latter determinant according to the elements of its last row (given by $i = t$). In the co-factor of a_{tj} $(j \geqq r+1)$ appear the elements $a_{kk} (k = 1, \cdots, \rho + 1)$, so that these co-factors are vanishing

---

* Read before the American Mathematical Society, March 22, 1913.

† Cf. Bôcher's Introduction to Higher Algebra, pp. 56–59.

‡ A short proof of different nature may be found in G. Kowalewski's Determinantentheorie, p. 122–4.

$M_r^{(\rho+1)}$. Call A_j the co-factor of a_{tj} $(j \leqq \rho)$. Since the co-factor of $a_{t_{\rho+r}}$ is $\pm M$,

$$a_{t1}A_1 + \cdots + a_{t\rho}A_\rho \pm a_{t_{\rho+1}}M = 0 \quad (t=r+1, \cdots, 2r-\rho).$$

A like equation with $t \leqq r$ holds since the sum of the products of the elements of a row by the co-factors of the elements of a different row is zero. Since $M \neq 0$, it follows (end of § 1) that

$$|a_{t1} \cdots a_{t_{\rho+1}}| = 0$$

for any $\rho + 1$ values $\leqq 2r - \rho$ of t. Interchange rows with columns and note that $a_{ij} = a_{ji}$. Thus

$$|a_{it}| = 0 \quad (i = 1, \cdots, \rho + 1;\ t \text{ with any } \rho + 1 \text{ values}).$$

Hence every $M_{\rho+1}$ formed from the first $\rho + 1$ rows of M is zero, in contradiction with $M \neq 0$. Thus every $M_r^{(\rho)} = 0$ and the induction is complete.

THE UNIVERSITY OF CHICAGO,
February, 1913.

NOTE ON THE RANK OF A SYMMETRICAL MATRIX.

By J. H. M. Wedderburn.

In the preceding paper, Professor Dickson has given a very simple proof of Kronecker's Theorem on the rank of a symmetrical matrix. The proof of this theorem which is given in this note, while not so elementary as Dickson's, shows very clearly the algebraic nature of the result.

Let $A = (a_{ij})$ be a symmetrical matrix of order n and rank r and

$$\varphi(x_1, \cdots, x_n) = \Sigma a_{ij} x_i x_j$$

the corresponding quadratic form. If the determinant of A is zero, there exists a point $(x_{11}, \cdots, x_{1n})$ such that

$$\sum_{j=1}^{n} a_{ij} x_{ij} = 0 \qquad (i = 1, 2, \cdots, n).$$

It is obvious geometrically, and not difficult to show algebraically, that there is an orthogonal matrix

$$B = (b_{ij}), \qquad |b_{ij}| \neq 0,$$

which transforms the point $(x_1, \cdots x_n)$ into $(1, 0, \cdots 0)$; and if the variables in the quadratic form are transformed by this orthogonal matrix, the matrix of the new form is $B'AB$ which is also symmetrical and has the same rank as A. From the way in which B was chosen, the coefficients in the first column of $B'AB$ are all zero and, as it is symmetrical, the same is true of the first row. The transformation therefore expresses φ in terms of $n - 1$ variables at most. This process may be repeated $n - r$ times the final result being a symmetrical matrix, $C'AC$, in which the first $n - r$ rows and columns are zero, while the determinant formed from the remaining elements is not zero. Since C, being the product of orthogonal matrices, is itself orthogonal, we have

$$C'AC = C^{-1}AC;$$

but the characteristic determinants $|A - \lambda|$ and $|C^{-1}AC - \lambda|$ are identical, and the coefficient of $(-1)^s \lambda^{n-s}$ in $|A - \lambda|$ is the sum of the principal minors of A of order s; therefore, since in $|C^{-1}AC - \lambda|$ this sum is zero if $s > r$ and not zero if $s = r$, we have the theorem:

In a symmetrical matrix of rank r the sum of the principal minors of order r is not zero.

It follows immediately that at least one principal minor of order r is not zero.

The same result is readily derived for skew symmetrical matrices and for matrices of the form $A + \lambda A'$.

Princeton University.

ON THE NUMERICAL FACTORS OF THE ARITHMETIC FORMS $\alpha^n \pm \beta^n$.*

By R. D. Carmichael.

Let $\alpha + \beta$ and $\alpha\beta$ be any two relatively prime integers (different from zero). Then α and β are roots of the quadratic equation

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0.$$

It is obvious that the numbers D_n and S_n,

$$D_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}, \qquad S_n = \alpha^n + \beta^n,$$

are integers, since they are expressed as rational integral symmetric functions of the roots of an algebraic equation with integral coefficients with leading coefficient unity. The principal object of the present paper is an investigation of the numerical factors of the numbers D_n and S_n. The case when α and β are roots of unity is excluded from consideration. (See § 2.)

The most valuable treatment of the questions connected with these numbers is that of Lucas.† The special case in which α and β are integers has been considered by Siebeck,‡ Birkhoff and Vandiver,§ Dickson,|| and Carmichael.¶

In Lucas's paper many results of interest and importance are obtained. The methods employed, however, are often indirect and cumbersome. In the present paper a direct and powerful method of treatment** is employed throughout; and in connection with the new results which are obtained many of Lucas's theorems are generalized and several errors†† in the statement of his conclusions are pointed out.

In § 1 several fundamental algebraic formulæ are obtained and a partial factorization of D_n and S_n is effected. In § 2 these algebraic formulæ are employed to derive numerous elementary properties of the integers

* Presented to the American Mathematical Society, December, 1912.

† American Journal of Mathematics, 1 (1878): 184–240, 289–321.

‡ Crelle's Journal, 33 (1846): 71–77.

§ *Annals of Mathematics*, (2) 5 (1904): 173–180.

|| American Mathematical Monthly, 12 (1905): 86–89.

¶ American Mathematical Monthly, 16 (1909): 153–159.

** Compare the method employed by Dickson in the paper already cited.

†† Compare the review of Lucas's paper in the Jahrbuch über die Fortschritte der Mathematik, 10 (1878): 134–136.

D_n and S_n relative to divisibility, and these properties are stated in explicit theorems.

In § 3 the important question of the appearance of a given prime factor in the sequence $D_1, D_2, D_3, \cdots$ is investigated. The principal results are contained in Theorems XII and XIII. Attention is called to the new number-theoretic functions introduced in connection with Theorem XIII and its corollary.

In § 4 a detailed study is made of the numerical factors of a set of numbers which are the values of an algebraic form $F_k(\alpha, \beta)$ which may be defined as that irreducible algebraic factor of $\alpha^k - \beta^k$ which is not a factor of any $\alpha^\nu - \beta^\nu$ for which $\nu < k$ (but see the definition in § 1). This investigation is fundamental in the study of the numbers D_n and S_n, and the results which are here obtained have important applications in the theory of numbers. Attention is called especially to Theorems XIV, XVI and XVIII.

In § 5 the theory of "characteristic factors" of F_n, D_n and S_n is developed.

In § 6 very simple proofs are given of certain special cases of Dirichlet's celebrated theorem concerning the prime terms of an arithmetical progression of integers; in particular, it is shown that there is an infinitude of prime numbers of each of the forms $4n + 1$, $4n - 1$, $6n + 1$, $6n - 1$.

In § 7 are given a number of theorems which are useful in the identification of large prime numbers. Among the results obtained the following two alone will be mentioned here: A necessary and sufficient condition that a given odd number p is prime is that an integer a exists such that

$$F_{p-1}(a, 1) \equiv 0 \bmod p;$$

a necessary and sufficient condition that $2^{2^n} + 1$, $n > 1$, is prime is that

$$3^{2^{2^n-1}} + 1 \equiv 0 \bmod 2^{2^n} + 1.$$

1. Notation. Fundamental Algebraic Formulæ.

Let

$$Q_n(x) = 0$$

be the algebraic equation whose roots are the primitive nth roots of unity without repetition, the coefficient of the highest power of x in $Q_n(x)$ being unity. The polynomial $Q_n(x)$ has all its coefficients integers; and it is of degree $\varphi(n)$, where $\varphi(n)$ denotes the number of integers not greater than n and prime to n.

From the theory* of the primitive roots of unity we have two formulæ

* See Bachmann's Kreistheilung, especially the third lecture.

which are fundamental for our purposes. Thus,

$$x^n - 1 = \prod_d Q_d(x), \tag{1}$$

where d ranges over all the divisors of n. Also,

$$Q_n(x) = \frac{(x^n - 1) \cdot \Pi(x^{n/p_ip_j} - 1) \cdots}{\Pi(x^{n/p_i} - 1) \cdot \Pi(x^{n/p_ip_jp_k} - 1) \cdots}, \tag{2}$$

where the p's denote the different prime factors of n and where the products denoted by Π extend over the combinations 2, 4, 6, $\cdots$ at a time of p_1, p_2, p_3, $\cdots$ in the numerator and over the combinations 1, 3, 5, $\cdots$ at a time in the denominator.

Let $\alpha + \beta$ and $\alpha\beta$ be any two relatively prime integers (different from zero); then α and β are the roots of the equation

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0$$

whose coefficients $\alpha + \beta$ and $\alpha\beta$ are any two relatively prime integers both of which are different from zero. We shall exclude the trivial case $\alpha = \beta = 1$. It is then clear that α and β cannot be equal.

Now $\alpha^n + \beta^n$ represents an integer for every value of n, since the function $\alpha^n + \beta^n$ is a symmetric polynomial in α and β and has integral coefficients. On the other hand the function $\alpha^n - \beta^n$ does not necessarily have an integral value. If, however, this number is divided by $\alpha - \beta$ the result is clearly an integer, since it may obviously be written as a rational integral symmetric function of α and β with integral coefficients. Accordingly, let us define the integers D_n and S_n, for every value of n, by the relations

$$D_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}, \qquad S_n = \alpha^n + \beta^n.$$

Then, obviously,

$$S_n = \frac{D_{2n}}{D_n},$$

so that a study of the factorization of the form D_n, for varying values of n, includes incidentally that of the form S_n. We shall therefore be interested primarily in the form D_n.

We define $F_k(\alpha, \beta)$ by the relation

$$F_k(\alpha, \beta) = \beta^{\phi(k)}Q_k(\alpha/\beta). \tag{3}$$

We shall now show that $F_k(\alpha, \beta)$ is an integer for every value of k except $k = 1$. The theorem is obviously true for $k = 2$; for,

$$F_2(\alpha, \beta) = \alpha + \beta.$$

Then suppose that k is greater than 2. Let ω be a primitive kth root of unity. Then evidently,

$$F_k(\alpha, \beta) = \beta^{\phi(k)} Q_k(\alpha/\beta) = \prod_{i=1}^{\phi(k)} (\alpha - \omega^{s_i}\beta), \tag{4}$$

where for $i = 1, 2, \cdots, \varphi(k)$, the s_i are the $\varphi(k)$ positive integers less than k and prime to k. Hence

$$F_k(\alpha, \beta) = \prod_{i=1}^{\phi(k)} (\alpha - \omega^{s_i}\beta)\omega^{k-s_i},$$

since

$$\omega^{k-s_j} \cdot \omega^{k-s_e} = 1$$

when

$$s_j + s_e = k$$

and the factors in the above equation obviously fall into pairs such that the sum of the s's in each pair is k. Hence we see readily that

$$F_k(\alpha, \beta) = \prod_{i=1}^{\phi(k)} (\alpha\omega^{k-s_i} - \beta) = \prod_{j=1}^{\phi(k)} (\beta - \omega^{s_j}\alpha),$$

where in the last member s_j is written for $k - s_i$. By comparing this equation with (4) we find that

$$F_k(\alpha, \beta) = F_k(\beta, \alpha);$$

that is, $F_k(\alpha, \beta)$ is symmetric with respect to α and β. But it is a polynomial in α and β with integral coefficients. Hence we conclude that

The number $F_k(\alpha, \beta)$ is an integer for every value of k except $k = 1$.

Now from (1) we have readily

$$D_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \prod_d{}' F_d(\alpha, \beta), \tag{5}$$

where d ranges over all the divisors of n except unity. *This important formula gives a (partial) factorization of the integer D_n.* Likewise, if ν is any divisor of n,

$$D_{n/\nu} = \prod_\delta{}' F_\delta(\alpha, \beta), \tag{6}$$

where δ ranges over all the divisors of n/ν except unity. If now we divide the first of these equations by the second, member for member, we have

$$\frac{D_n}{D_{n/\nu}} = \alpha^{n(\nu-1)/\nu} + \alpha^{n(\nu-2)/\nu}\beta^{n/\nu} + \cdots + \alpha^{n/\nu}\beta^{n(\nu-2)/\nu} + \beta^{n(\nu-1)/\nu} = \prod_k F_k(\alpha, \beta), \tag{7}$$

where k ranges over all the divisors of n which are not at the same time divisors of n/ν.

From (2) we obtain readily the equation

$$F_n(\alpha, \beta) = \frac{(\alpha^n - \beta^n) \cdot \Pi(\alpha^{n/p_ip_j} - \beta^{n/p_ip_j}) \cdots}{\Pi(\alpha^{n/p_i} - \beta^{n/p_i}) \cdot \Pi(\alpha^{n/p_ip_jp_k} - \beta^{p_ip_jp_k}) \cdots}, \tag{8}$$

where the factors denoted by Π extend over the combinations 2, 4, 6, $\cdots$ at a time of $p_1, p_2, \cdots$ in the numerator and over the combinations 1, 3, 5, $\cdots$ at a time in the denominator. The total number of factors in the numerator of this equation is the same as that in the denominator; for, obviously, the first of these numbers is the sum of the positive terms and the second is the sum of the negative terms in the expansion of $(1 - 1)^r$ by the binomial formula, r being the number of different prime factors of n. Hence, dividing each of these factors in both numerator and denominator by $\alpha - \beta$, we have

$$F_n(\alpha, \beta) = \frac{D_n \cdot \Pi D_{n/p_ip_j} \cdots}{\Pi D_{n/p_i} \cdot \Pi D_{n/p_ip_jp_k} \cdots}, \tag{9}$$

where the products denoted by Π have a meaning similar to that above.

Let p be any prime factor of n and write

$$n = \nu p^a$$

where the exponent a is so chosen that ν is an integer which is not divisible by p. Consider the factors in the second member of (9) into which p does not enter explicitly; from (9) itself it is clear that these factors alone have the value $F_\nu(\alpha^{p^a}, \beta^{p^a})$. In the same way we see that the factors into which p enters explicitly have the value $1/F_\nu(\alpha^{p^{a-1}}, \beta^{p^{a-1}})$. Hence

$$F_n(\alpha, \beta) = F_\nu(\alpha^{p^a}, \beta^{p^a}) \div F_\nu(\alpha^{p^{a-1}}, \beta^{p^{a-1}}). \tag{10}$$

Since

$$F_1(\alpha, \beta) = \alpha - \beta,$$

equation (10) may be used as a recursion formula for determining $F_n(\alpha, \beta)$. For $n \leqq 36$, Sylvester's table* of cyclotomic functions may conveniently be employed for finding $F_n(\alpha, \beta)$.

In passing we note without demonstration that (10) may be proved directly and then be employed for the derivation of (9).†

If, now, in equation (7) we replace n by $2n$, give to ν the value 2 and remember that

$$\frac{D_{2n}}{D_n} = S_n,$$

we have

$$S_n = \alpha^n + \beta^n = \prod_k F_k(\alpha, \beta), \tag{11}$$

* American Journal of Mathematics, 2 (1879): 367–368.

† Compare Dickson, l. c., p. 86.

where k runs over all those divisors of $2n$ which contain 2 to the same power as $2n$ itself. *This important formula gives a (partial) factorization of the integer S_n.*

Let ν be any *odd* divisor of n; then, writing n/ν for n in (11) we have

$$(12) \qquad S_{n/\nu} = \prod_k F_k(\alpha, \beta),$$

where k runs over all those divisors of $2n/\nu$ which contain 2 to the same power as $2n/\nu$ itself. Dividing (11) by (12), member for member, we have

$$(13) \qquad \frac{S_n}{S_{n/\nu}} = \prod_k F_k(\alpha, \beta), \qquad \nu \text{ odd},$$

where k runs over all those divisors of $2n$ which contain 2 to the same power as $2n$ itself and which do not divide $2n/\nu$.

2. General Properties of the Integers D_n and S_n Relative to Divisibility.

In view of the fact that a rational integral symmetric function of α, β with integral coefficients is an integer we have readily the two equations

$$(\alpha + \beta)^n = \alpha^n + \beta^n + \alpha\beta I_1 = S_n + \alpha\beta I_1,$$

$$D_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} + \beta^{n-1} + \alpha\beta I_2 = S_{n-1} + \alpha\beta I_2,$$

where I_1 and I_2 are integers. Since $\alpha\beta$ and $\alpha + \beta$ are relatively prime integers it follows from the first of these equations that S_n is prime to $\alpha\beta$ for every value of n. Then from the second of the equations we conclude that D_n is likewise prime to $\alpha\beta$ for every value of n. Hence we have the following theorem:

THEOREM I. *The integers D_n and S_n are both prime to $\alpha\beta$.*

This theorem enables us to dispose of an exceptional case; namely, when $D_m = 0$ for some value of m. In this case $\alpha^m = \beta^m$ and hence

$$S_m = 2\alpha^m.$$

But S_m is prime to $\alpha\beta$ and hence to $\alpha^m\beta^m$. These two results agree only when

$$\alpha^m = \beta^m = \pm 1,$$

so that in this case α and β are both roots of unity. It is easy to see that S_k can assume no other value than $-2, -1, 0, 1, 2$; for

$$|S_k| \leqq |\alpha^k| + |\beta^k| = 2.$$

Now

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \text{integer};$$

and hence

$$|\alpha - \beta| \overline{\overline{>}} 1,$$

since $\alpha \neq \beta$. Therefore

$$|D_k| \leqq |\alpha^k - \beta^k| \leqq |\alpha^k| + |\beta^k| = 2,$$

so that D_k can take only the values $-2, -1, 0, 1, 2$. A corresponding discussion can be made when $S_m = 0$ for some value of m, and with like results. The cases $D_m = 0$ for some m and $S_m = 0$ for some m are therefore both trivial. They arise when and only when α and β are roots of unity. Hence in what follows *we shall exclude from consideration the case in which α and β are roots of unity.* Then *D_m and S_m are always different from zero.*

Now

$$(\alpha^n + \beta^n)^2 - (\alpha^n - \beta^n)^2 = 4\alpha^n\beta^n,$$

and hence

$$S_n^2 - (\alpha - \beta)^2 D_n^2 = 4\alpha^n\beta^n.$$

It is clear that $(\alpha - \beta)^2$ is an integer. Then from the above equation it follows that any common divisor of S_n^2 and D_n^2 must be a divisor of $4\alpha^n\beta^n$; but by Theorem I such a divisor is prime to $\alpha\beta$. Hence it is a divisor of 4. Therefore, either D_n and S_n are relatively prime or they have the greatest common divisor 2. That both of these cases may arise is shown by the following examples:

(1) $\alpha = 2, \beta = 1$. D_n and S_n have not the common divisor 2 and hence are relatively prime;

(2) $\alpha = 3, \beta = 1$. D_n and S_n have the common factor 2 if n is even.

Hence we have the following theorem:*

THEOREM II. *The integers D_n and S_n either are relatively prime or have the greatest common divisor* 2.

We shall now determine the character of D_n and S_n relative to divisibility by 2. From Theorem I it follows that both of them are odd when $\alpha\beta$ is even. Hence we have to treat further only the case when $\alpha\beta$ is odd. This will separate further into two cases according as $\alpha + \beta$ is odd or even. We start from the recurrence formulæ

$$(14)\qquad \begin{aligned} D_{n+2} - (\alpha + \beta)D_{n+1} + \alpha\beta D_n &= 0, \\ S_{n+2} - (\alpha + \beta)S_{n+1} + \alpha\beta S_n &= 0, \end{aligned}$$

which are readily verified by substituting for D_k and S_k, $k = n, n + 1, n + 2$, their values in terms of α and β. Since for the present discussion $\alpha\beta$ is odd, we have from (14)

$$D_{n+2} \equiv D_n, \qquad S_{n+2} \equiv S_n \mod 2$$

or

$$D_{n+2} \equiv D_{n+1} + D_n, \qquad S_{n+2} \equiv S_{n+1} + S_n \mod 2$$

according as $\alpha + \beta$ is even or odd.

* Lucas (l. c., p. 200) states inaccurately that D_n and S_n are relatively prime.

Now $D_1 = 1$ and $D_2 = \alpha + \beta$. Hence from the above congruences which involve D_n we see readily that when $\alpha + \beta$ is even D_n is even or odd according as n is even or odd; and that when $\alpha + \beta$ is odd, D_n is even or odd according as n is or is not a multiple of 3.

We treat the number S_n in a similar manner. We have

$$S_1 = \alpha + \beta, \qquad S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta.$$

Hence, if $\alpha + \beta$ is odd both S_1 and S_2 are odd; and if $\alpha + \beta$ is even both S_1 and S_2 are even. Therefore from the above congruences involving S_n we conclude readily that if $\alpha + \beta$ is even S_n is even for all values of n; and that if $\alpha + \beta$ is odd S_n is even or odd according as n is or is not a multiple of 3.

Collecting these results we have the following theorem:

THEOREM III. *If $\alpha\beta$ is even both D_n and S_n are odd. If $\alpha\beta$ is odd and $\alpha + \beta$ is even, then S_n is even for all values of n while D_n is even or odd according as n is even or odd. If both $\alpha\beta$ and $\alpha + \beta$ are odd then D_n and S_n are both even or both odd according as n is or is not a multiple of 3.*

From the properties of symmetric functions of the roots of an algebraic equation and the algebraic divisibility of D_n by D_ν when ν is a divisor of n, it follows immediately that the integer D_n is divisible by the integer D_ν when ν is a divisor of n. This is also an immediate consequence of equation (7); and the latter equation in general states more than this, that is, it gives a partial factorization of the integer D_n/D_ν. Thus we have the following theorem:

THEOREM IV. *If ν is a divisor of n then D_ν is a divisor of D_n and we have*

$$\frac{D_n}{D_\nu} = \prod_k F_k(\alpha, \beta),$$

where k ranges over all those divisors of n which are not at the same time divisors of ν.

For $\nu = 1$ this theorem gives a partial factorization of D_n, since $D_1 = 1$. In the preceding section we proved that the quantities $F_k(\alpha, \beta)$ have integer values.

By the aid of equation (13) the following theorem may be demonstrated:

THEOREM V. *If ν is a divisor of n such that n/ν is odd then S_n is divisible by S_ν and we have*

$$\frac{S_n}{S_\nu} = \prod_k F_k(\alpha, \beta),$$

where k runs over all those divisors of $2n$ which contain 2 to the same power as $2n$ itself and which do not divide 2ν.

From the identity

$$(\alpha^m - \beta^m)(\alpha^n + \beta^n) - (\alpha^n - \beta^n)(\alpha^m + \beta^m) = 2\alpha^n\beta^n(\alpha^{m-n} - \beta^{m-n}), \qquad m > n,$$

we have readily

$$D_mS_n - D_nS_m = 2\alpha^n\beta^nD_{m-n}. \tag{15}$$

From this equation and the fact that D_m and D_n are prime to $\alpha\beta$ it follows that every common *odd* divisor of D_m and D_n is also a divisor of D_{m-n}; whence we conclude readily that every common odd divisor of D_m and D_n is a divisor of D_ν where ν is the greatest common divisor of m and n. But according to Theorem IV D_ν is a divisor of D_m and D_n. Hence the greatest common divisor of D_m and D_n is D_ν provided that either D_m/D_ν or D_n/D_ν is odd. This latter fact we shall now prove by aid of Theorems I and III.

We have

$$\frac{D_m}{D_\nu} = \frac{\alpha^m - \beta^m}{\alpha^\nu - \beta^\nu} = \frac{\bar{\alpha}^{m/\nu} - \bar{\beta}^{m/\nu}}{\bar{\alpha} - \bar{\beta}},$$

if we replace α^ν, β^ν by $\bar{\alpha}$, $\bar{\beta}$. The last member of the above equation we denote by $\bar{D}_{m/\nu}$. We define $\bar{D}_{n/\nu}$ in a similar manner. It follows from Theorem I that $\alpha^\nu\beta^\nu$ and $\alpha^\nu + \beta^\nu$ are relatively prime. They are both different from zero. That is, $\bar{\alpha}\bar{\beta}$ and $\bar{\alpha} + \bar{\beta}$ are relatively prime integers both of which are different from zero. Hence we may apply Theorem III to $\bar{D}_{m/\nu}$ and $\bar{D}_{n/\nu}$. If $\bar{\alpha}\bar{\beta}$ is even both of these numbers are odd. If $\bar{\alpha}\bar{\beta}$ is odd and $\bar{\alpha} + \bar{\beta}$ is even one of the numbers $\bar{D}_{m/\nu}$ and $\bar{D}_{n/\nu}$ is odd; for either m/ν or n/ν is odd, since ν is the greatest common divisor of m and n. Likewise, if $\bar{\alpha}\bar{\beta}$ and $\bar{\alpha} + \bar{\beta}$ are both odd then one of the numbers $\bar{D}_{m/\nu}$, and $\bar{D}_{n/\nu}$, is odd; for either m/ν or n/ν is not divisible by 3, since ν is the greatest common divisor of m and n. Hence $\bar{D}_{m/\nu}$ and $\bar{D}_{n/\nu}$ have not the common factor 2.

Remembering that $\bar{D}_{m/\nu} = D_m/D_\nu$ and $\bar{D}_{n/\nu} = D_n/D_\nu$ and making use of the results of the last two paragraphs we have the theorem:*

THEOREM VI. *The greatest common divisor of D_m and D_n is D_ν where ν is the greatest common divisor of m and n.*

Since $D_1 = 1$ we have at once the following corollary:

COROLLARY. *The integers D_m and D_n are relatively prime when m and n are relatively prime.*

The example

$$S_6(2, 1) = 2^6 + 1 = 5.13, \qquad S_4 = 2^4 + 1 = 17, \qquad S_2 = 2^2 + 1 = 5$$

shows at once that the greatest common divisor of S_m and S_n is not always

---

* The part of this theorem which applies to the odd divisors of D_m and D_n is due to Lucas (l. c., p. 206).

S_ν where ν is the greatest common divisor of m and n. If, however, m/ν and n/ν are both odd this simple law obtains, as we now show. In this case it follows from Theorem V that S_ν is a common divisor of S_m and S_n. Now

$$D_{2m} = S_m D_m$$

and

$$D_{2n} = S_n D_n,$$

whence we conclude by aid of Theorem VI that the greatest common divisor of S_m and S_n is a factor of $D_{2\nu}$. Now

$$D_{2\nu} = S_\nu D_\nu$$

and hence we have only to examine what factors D_ν has in common with S_m and S_n. Now D_ν is a factor of D_m, and D_m and S_m have the greatest common divisor 1 or 2. Hence D_ν has with S_m and S_n the greatest common divisor 1 or 2. Therefore S_m and S_n have the greatest common divisor S_ν or $2S_\nu$; and in the next two paragraphs we show that the latter case does not arise.

To prove that the greatest common divisor under consideration is not $2S_\nu$ it is sufficient to show that either S_m/S_ν or S_n/S_ν is odd. This follows at once from Theorem III if $\alpha\beta$ is even; for then S_m and S_n are odd. In general

$$\frac{S_m}{S_\nu} = \frac{\alpha^m + \beta^m}{\alpha^\nu + \beta^\nu} = \frac{\bar{\alpha}^{m/\nu} + \bar{\beta}^{n/\nu}}{\bar{\alpha} + \bar{\beta}},$$

if $\alpha^\nu = \bar{\alpha}$ and $\beta^\nu = \bar{\beta}$. Denote the last numerator above by $\bar{S}_{m/\nu}$ and define $\bar{S}_{n/\nu}$ in a similar way. Then Theorem III is applicable to $\bar{S}_{m/\nu}$ and $\bar{S}_{n/\nu}$. Now either m/ν or n/ν is prime to 3, and hence one of the numbers $\bar{S}_{m/\nu}$ and $\bar{S}_{n/\nu}$ is odd if $\bar{\alpha}\bar{\beta}$ and $\bar{\alpha} + \bar{\beta}$ are both odd, that is, if $\alpha\beta$ and $\alpha + \beta$ are both odd. In this case, then, one at least of the numbers S_m/S_ν and S_n/S_ν is odd.

Let us next consider the case in which $\alpha\beta$ is odd and $\alpha + \beta$ is even; say that $\alpha + \beta$ is an odd multiple of 2^k. Then, since

$$S_1 = \alpha + \beta$$

and

$$S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta,$$

it is easy to see that S_1 and S_2 are odd multiples of 2^k and 2 respectively. By means of the second recursion formula (14) one sees that in general S_n is an odd multiple of 2^k or of 2 according as n is odd or even. Hence in this case S_m/S_ν and S_n/S_ν are both odd, since m and ν and likewise n and ν are both odd or both even.

Thus we have the following theorem:

THEOREM VII. *If ν is the greatest common divisor of m and n, and m/ν and n/ν are both odd, then the greatest common divisor of S_m and S_n is S_ν.*

We turn now to an interesting theorem of a different character, namely:

THEOREM VIII. *Let $m_1, m_2, \cdots, m_s$ and $n_1, n_2, \cdots, n_r$ be two sets of positive integers which have the property that any positive integer d, different from unity, which is a factor of (just) t integers of the second set is also a factor of at least t integers of the first set; then the number*

$$\frac{D_{m_1} \cdot D_{m_2} \cdot \cdots \cdot D_{m_s}}{D_{n_1} \cdot D_{n_2} \cdot \cdots \cdot D_{n_r}}$$

is an integer.

This theorem is an immediate consequence of the (partial) factorization of D_n given in equation (5).

COROLLARY I. *The product of any n consecutive terms of the sequence $D_1, D_2, D_3, \cdots$ is divisible by the product of the first n terms.**

COROLLARY II. *The number*

$$\frac{D_1 D_2 \cdots D_{n_1+n_2+\cdots+n_k}}{(D_1 D_2 \cdots D_{n_1})(D_1 D_2 \cdots D_{n_2}) \cdots (D_1 D_2 \cdots D_{n_k})}$$

is an integer.

This result is analogous to the theorem that the polynomial coefficient

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!\, n_2! \cdots n_k!}$$

is an integer.

Let m and n be any two relatively prime positive integers and suppose that the positive integer d $(d \neq 1)$ is a divisor of s integers of the set 1, 2, $\cdots$, m and of t integers of the set 1, 2, $\cdots$, n. Then d is obviously a divisor of at least $s + t$ integers of the set 1, 2, $\cdots$, $m + n - 1$. In view of this fact Theorem VIII yields the further corollary:

COROLLARY III. *If m and n are any two relatively prime positive integers, then the number*

$$\frac{D_1 D_2 \cdots D_{m+n-1}}{(D_1 D_2 \cdots D_m)(D_1 D_2 \cdots D_n)}$$

is an integer.

This theorem is analogous to that which asserts that

$$\frac{(m + n - 1)!}{m!\, n!}$$

is an integer, provided that m and n are relatively prime.

---

* The result contained in this corollary is due to Lucas, who gave, however, a very different proof of it (Lucas, l. c., p. 203).

Similarly one may prove an extended analogue of the theorem which states that

$$\frac{(km_1)!\,(km_2)!\cdots(km_k)!}{m_1!\,m_2!\cdots m_k!\,(m_1+m_2+\cdots+m_k)!},\qquad k \gtreqless 2,$$

is an integer, namely:

COROLLARY IV. *The number*

$$\frac{(D_1D_2\cdots D_{km_1})(D_1D_2\cdots D_{km_2})\cdots(D_1D_2\cdots D_{km_k})}{(D_1D_2\cdots D_{m_1})^{k-1}\cdots(D_1D_2\cdots D_{m_k})^{k-1}(D_1D_2\cdots D_{m_1+m_2+\cdots+m_k})}$$

is an integer.

Just as equation (5) was used in the demonstration of Theorem VIII we may employ equation (11) to prove the following theorem:

THEOREM IX. *Let $m_1, m_2, \cdots, m_s$ and $n_1, n_2, \cdots, n_r$ be two sets of positive integers such that every positive integer d which is a factor of (just) t of the numbers $n_1, n_2, \cdots, n_r$ with odd quotient is also a factor of at least t of the numbers $m_1, m_2, \cdots, m_s$ with odd quotient. Then the number*

$$\frac{S_{m_1}\cdot S_{m_2}\cdot\cdots\cdot S_{m_s}}{S_{n_1}\cdot S_{n_2}\cdot\cdots\cdot S_{n_r}}$$

is an integer.

COROLLARY. *The product of any $2n-1$ consecutive terms of the sequence $S_1, S_3, S_5, \cdots$ is divisible by the product of the first n terms.*

If m is any integer and q is any odd prime, it is obvious that there exist integers

$$a_1, a_2, \cdots, a_s, \qquad s = \frac{q-1}{2},$$

dependent on q alone, such that

$$\alpha^{mq} - \beta^{mq} = (\alpha^m - \beta^m)^q + a_1\alpha^m\beta^m(\alpha^m - \beta^m)^{q-2} + a_2\alpha^{2m}\beta^{2m}(\alpha^m - \beta^m)^{q-4} + \cdots + a_s\alpha^{sm}\beta^{sm}(\alpha^m - \beta^m);$$

whence

$$(16)\quad D_{mq} = (\alpha-\beta)^{q-1}D_m{}^q + a_1(\alpha-\beta)^{q-3}\alpha^m\beta^m D_m{}^{q-2} + \cdots + a_s\alpha^{sm}\beta^{sm}D_m.$$

Let us evaluate a_s. Since it is independent of α, β and m, we may choose any convenient values for these numbers. Then put $m = 1$, $\beta = 1$, $\alpha = r + 1$, where r is a positive integer to be chosen at convenience. Then from (16) we have

$$\frac{(r+1)^q - 1}{r} \equiv a_s(r+1)^s \bmod r.$$

If we suppose r to be a prime number different from q we see that a_s is not divisible by r. If we put $r = q^2$ it follows that a_s is divisible by q but not by q^2. Hence $a_s = q$.

Suppose now that D_m is divisible by p^λ, $\lambda \neq 0$, and by no higher power of p, p being a prime number; then from (16), since $a_s = q$, we have

$$D_{mq} \equiv q\alpha^{sm}\beta^{sm}D_m \bmod p^{3\lambda}. \tag{17}$$

From this congruence it follows that $p^{\lambda+1}$ is the highest power of p contained in D_{mp}, provided that p is odd, and that p^λ is the highest power of p contained in D_{mq} when q is an odd prime different from p. We enquire further: What is the highest power of p contained in D_{2m}? We have $D_{2m} = D_m S_m$. In Theorem III we have seen that D_m and S_m have no common odd factor (different from unity). Hence, if p is an odd prime the highest power of p contained in D_{2m} is p^λ. If p is even, so that D_m is divisible by 2, it follows from Theorem III that S_m is divisible by 2. Then it follows from Theorem II that D_m and S_m have the highest common factor 2. Hence in this case D_{2m} contains $2^{\lambda+1}$; and it contains no higher power of 2 unless $\lambda = 1$.

These results lead to the following theorem:

THEOREM X. *If for $\lambda > 0$, $p^\lambda \neq 2$, p^λ is the highest power of a prime p contained in D_m then the highest power of p contained in $D_{m\mu p^a}$ is $p^{a+\lambda}$, μ being any number prime to p. If $p^\lambda = 2$, then $D_{m\mu 2^a}$ contains the factor 2^{a+1} and $D_{m\mu}$ is an odd multiple of 2.**

Suppose that S_m is divisible by p^λ, $\lambda > 0$, but by no higher power of the odd prime p. Then D_{2m} contains p^λ and no higher power of p, since

$$D_{2m} = D_m S_m$$

and D_m and S_m have no common odd prime factor. Therefore, according to the preceding theorem, $D_{2m\mu p^a}$, or $D_{m\mu p^a} \cdot S_{m\mu p^a}$, μ being prime to p, contains $p^{a+\lambda}$ and no higher power of p. Moreover $D_{m\mu p^a}$ and $S_{m\mu p^a}$ do not have a factor p in common. Hence one of these numbers contains $p^{a+\lambda}$ and no higher power of p while the other is prime to p. Since D_{2m} is a divisor of $D_{m\mu p^a}$ if μ is even, we see that $D_{m\mu p^a}$ contains $p^{a+\lambda}$ when μ is even. When μ is odd S_m is a factor of $S_{m\mu p^a}$ and hence in this case $S_{m\mu p^a}$ contains the factor $p^{a+\lambda}$.

Thus we have the following theorem:

THEOREM XI. *If p^λ, $\lambda > 0$, is the highest power of an odd prime p contained in S_m and μ is a number prime to p; then if μ is even $D_{m\mu p^a}$ is divisible by $p^{a+\lambda}$ and by no higher power of p and $S_{m\mu p^a}$ is prime to p, while if μ is odd $D_{m\mu p^a}$ is prime to p and $S_{m\mu p^a}$ is divisible by $p^{a+\lambda}$ and by no higher power of p.*

---

* The special case of this theorem in which $\mu = 1$ is given by Lucas (l. c., p. 210), but Lucas failed to notice the exceptional character of the case when $p^\lambda = 2$.

3. On the Appearance of a Given Prime Factor in the Sequence $D_1, D_2, D_3, \cdots$.

If it is known that a prime number p is a factor of D_m, theorems in the preceding section enable us to say how p enters into $D_{m\mu p^a}$. In the present section we show that any given prime p, which is not a factor of $\alpha\beta$, is a factor of a certain definite number of the sequence $D_1, D_2, D_3, \cdots$; we also carry out other related investigations. We have need of two lemmas, as follows:

LEMMA I. *If $S(\alpha^p, \beta^p)$ is any rational integral symmetric function of α^p, β^p with integral coefficients, then*

$$S(\alpha^p, \beta^p) \equiv S(\alpha, \beta) \bmod p,$$

p being a prime number.

The proof is not difficult. From Fermat's theorem it follows that

$$\alpha^p\beta^p \equiv \alpha\beta \bmod p, \tag{18}$$

since $\alpha\beta$ is an integer. Likewise

$$(\alpha + \beta)^p \equiv \alpha + \beta \bmod p.$$

But by the aid of the binomial formula we see that

$$(\alpha + \beta)^p \equiv \alpha^p + \beta^p \bmod p,$$

since the binomial coefficients for the prime exponent p are all multiples of p and $(\alpha + \beta)^p - (\alpha^p + \beta^p)$ is therefore clearly p times a polynomial which is symmetric in α, β and has integral coefficients; that is, $(\alpha + \beta)^p - (\alpha^p + \beta^p)$ is p times an integer. Hence

$$\alpha^p + \beta^p \equiv \alpha + \beta \bmod p. \tag{19}$$

But, since α^p and β^p are roots of the equation

$$x^2 - (\alpha^p + \beta^p)x + \alpha^p\beta^p = 0,$$

it is a consequence of the theory of symmetric functions of the roots of an algebraic equation that $S(\alpha^p, \beta^p)$ can be expressed in the form

$$S(\alpha^p, \beta^p) = P(\alpha^p + \beta^p, \alpha^p\beta^p),$$

where P is a polynomial in $\alpha^p + \beta^p$, $\alpha^p\beta^p$ with integral coefficients. From (18) and (19) it follows that

$$P(\alpha^p + \beta^p, \alpha^p\beta^p) \equiv P(\alpha + \beta, \alpha\beta) \bmod p.$$

But

$$P(\alpha + \beta, \alpha\beta) = S(\alpha, \beta)$$

and therefore

$$S(\alpha^p, \beta^p) \equiv S(\alpha, \beta) \bmod p,$$

as was to be proved.

If m is any integer and q is an odd prime, we have an identity of the form

$$(\alpha^m - \beta^m)^q = (\alpha^{qm} - \beta^{qm}) - q\alpha^m\beta^m(\alpha^{m(q-2)} - \beta^{m(q-2)}) + \cdots;$$

whence it follows that

$$(\alpha - \beta)^{q-1}D_m{}^q = D_{mq} + qI,$$

where I is an integer. Hence

$$D_{mq} \equiv (\alpha - \beta)^{q-1}D_m \bmod q.$$

Hence,

Lemma II. *If m is any integer and q is any odd prime, we have*

$$D_{mq} \equiv (\alpha - \beta)^{q-1}D_m \bmod q.$$

In particular,

$$D_{q^a} \equiv (\alpha - \beta)^{q-1}D_{q^{a-1}} \equiv \cdots \equiv (\alpha - \beta)^{a(q-1)}D_1 \bmod q.$$

Hence, since $D_1 = 1$, it follows that D_{q^a} is divisible by q when and only when $(\alpha - \beta)^2$ is divisible by q.

Theorem III gives exact information concerning the divisibility of D_n and S_n by 2. We shall now consider the question of the entrance of an odd prime factor q. If q is a factor of $\alpha\beta$ it follows from Theorem I that it does not divide either D_n or S_n. If it is a factor of $(\alpha - \beta)^2$ then it divides D_q, as we readily see from Lemma II. In what follows we shall consider the divisibility of D_n and S_n by an odd prime p which is not a divisor of either $(\alpha - \beta)^2$ or $\alpha\beta$.

If in equation (15) we put $m = p$ and $n = 1$ we have

$$D_pS_1 - D_1S_p = 2\alpha\beta D_{p-1},$$

or

$$(\alpha + \beta)D_p - S_p = 2\alpha\beta D_{p-1}.$$

From Lemma II it follows that

$$D_p \equiv (\alpha - \beta)^{p-1} \bmod p,$$

and from Lemma I that

$$S_p \equiv \alpha + \beta \bmod p.$$

Hence from the last equation we have

$$(\alpha + \beta)(\alpha - \beta)^{p-1} - (\alpha + \beta) \equiv 2\alpha\beta D_{p-1} \bmod p.$$

Now $(\alpha - \beta)^2$ is an integer; and therefore it follows from Fermat's theorem that

$$(\alpha - \beta)^{p-1} \equiv \pm 1 \bmod p.$$

Hence from the above congruence we have the two cases

$$D_{p-1} \equiv 0 \bmod p \quad \text{if} \quad (\alpha - \beta)^{p-1} \equiv 1 \bmod p,$$

$$\alpha\beta D_{p-1} \equiv -(\alpha + \beta) \bmod p \quad \text{if} \quad (\alpha - \beta)^{p-1} \equiv -1 \bmod p.$$

Now it is easy to verify that

$$D_{p+1} - (\alpha + \beta)D_p + \alpha\beta D_{p-1} = 0;$$

and hence we see that

$$D_{p+1} \equiv 0 \bmod p \quad \text{if} \quad (\alpha - \beta)^{p-1} \equiv -1 \bmod p.$$

Therefore we have the following theorem:*

THEOREM XII. *An odd prime p which does not divide either $(\alpha - \beta)^2$ or $\alpha\beta$ is a factor of D_{p-1} or of D_{p+1} according as $(\alpha - \beta)^{p-1}$ is congruent to $+1$ or to -1 modulo p.*

Obviously, if $\alpha - \beta$ is an integer (that is, if α and β are integers) we have always that D_{p-1} is divisible by p.

By means of Theorems X and XII we are now to prove a result of fundamental importance. In order to be able to state this result succinctly we shall employ a number-theory function $\lambda_{rs}(n)$ which we define below. It is convenient at the same time to define a second function $\varphi_{rs}(n)$ which is intimately related to $\lambda_{rs}(n)$.

Let rs and $r + s$ be any two integers; that is, let r and s be the roots of any quadratic equation of the form

$$x^2 - ux + v = 0$$

where u and v are integers. When p is an odd prime we define the symbol $\left(\frac{r,\, s}{p}\right)$ by the congruence

$$(r - s)^{p-1} \equiv \left(\frac{r,\, s}{p}\right) \bmod p,$$

it being understood that $\left(\frac{r,\, s}{p}\right)$ is the residue of least absolute value; whence $\left(\frac{r,\, s}{p}\right) = 0, +1,$ or -1 according as $(r - s)^2$ is divisible by p, is a quadratic residue of p, or is a quadratic non-residue of p. The symbol $\left(\frac{r,\, s}{2}\right)$ is defined thus:

$$\left(\frac{r,\, s}{2}\right) = 1, \text{ if } rs \text{ is even;}$$

$$\left(\frac{r,\, s}{2}\right) = 0, \text{ if } rs \text{ is odd and } r + s \text{ is even;}$$

$$\left(\frac{r,\, s}{2}\right) = -1, \text{ if } rs \text{ and } r + s \text{ are both odd.}$$

* This theorem is due to Lucas (l. c., pp. 290, 296, 297). Lucas's proof, however, is different from that above.

Then if

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where $p_1, p_2, \cdots, p_k$ are the different prime factors of n, we define $\varphi_{rs}(n)$ by the equation

$$\varphi_{rs}(n) = \prod_{i=1}^{k} p_i^{a_i-1}\left[p_i - \left(\frac{r, s}{p_i}\right)\right].$$

This function is similar to one introduced by Lucas, l. c., p. 300. It is, however, somewhat more general. For $r = 2$ and $s = 1$ we have

$$\varphi_{21}(n) = \varphi(n),$$

where $\varphi(n)$ is Euler's φ-function of n. The function introduced by Lucas does not have this interesting property of including the φ-function as a special case.

The functional value $\lambda_{rs}(n)$ is defined to be the least common multiple of the numbers

$$p_i^{a_i-1}\left[p_i - \left(\frac{r, s}{p_i}\right)\right], \qquad i = 1, 2, \cdots, k.$$

It is obvious that $\lambda_{rs}(n)$ is a divisor of $\varphi_{rs}(n)$.

The functions $\varphi_{rs}(n)$ and $\lambda_{rs}(n)$ have several important properties; but this is not an appropriate place to develop them in full.

The fundamental theorem to be proved may now be stated as follows:

THEOREM XIII. *If the number n,*

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where $p_1, p_2, \cdots, p_k$ are the different prime factors of n, is prime to $\alpha\beta$ and if

$$\lambda = \lambda_{\alpha\beta}(n),$$

we have

$$D_\lambda \equiv 0 \bmod n.$$

To prove this theorem it is sufficient to show that D_λ contains the factor $p_i^{a_i}$ where i is any number of the set $1, 2, \cdots, k$. This follows at once from previous results. For, λ is a multiple of t_i,

$$t_i = p_i^{a_i-1}\left[p_i - \left(\frac{\alpha, \beta}{p_i}\right)\right] = p_i^{a_i-1}k_i,$$

say. From Theorems XII and III and the remark following Lemma II we see that D_{k_i} is in every case divisible by p_i; and hence from X that D_λ is divisible by $p_i^{a_i}$.

COROLLARY.* *If $\varphi = \varphi_{\alpha\beta}(n)$, then $D_\varphi \equiv 0 \bmod n$.*

---

* This corollary is essentially the same as a certain fundamental result due to Lucas, l. c., p. 300. It should be noted that Lucas's statement of this theorem is not entirely accurate.

In connection with these simple theorems concerning the divisors of the numbers in the sequence $D_1, D_2, \cdots$, it should be noticed that no laws of corresponding simplicity obtain in the case of the sequence $S_1, S_2, \cdots$. We have seen that an odd prime p which does not divide either $(\alpha - \beta)^2$ or $\alpha\beta$ is a factor of D_{p-1} or of D_{p+1}. But in the case of the sequence $S_1, S_2, \cdots$ it often happens that a given prime number is not a factor of any term. Thus 7 is not a factor of $S_n(2, 1), \equiv 2^n + 1$, for any value of n. More generally, suppose that D_k, where k is odd, has an odd prime factor p while p is not a divisor of any D_ν for ν less than k. From Theorem VI it follows that D_m is divisible by p when and only when m is a multiple of k. If we suppose that p is a divisor of S_n for any given value of n we shall be led to a contradiction. For, since $D_{2n} = D_nS_n$, D_{2n} is divisible by p; and therefore $2n$ is a multiple of k. But k is odd, and hence n is a multiple of k. Therefore D_n is divisible by p; and D_n and S_n have the common odd prime factor p, which is impossible. Hence, *an odd prime number p which divides D_k, where k is odd, and does not divide any D_ν for ν less than k, is not a factor of any S_n.*

4. On the Numerical Factors of the Forms $F_k(\alpha, \beta)$.

We have already seen that the numbers $F_k(\alpha, \beta)$ are of fundamental importance in the factorization of D_n and S_n. We turn therefore to a detailed treatment of these numbers.

Let us suppose that

$$F_\nu(\alpha, \beta) \equiv 0 \bmod p, \qquad \nu > 1,$$

and that ν is not a multiple of the prime number p. Suppose that k is a subscript for which

$$F_k(\alpha, \beta) \equiv 0 \bmod p.$$

Now* F_ν and F_k are divisors of D_ν and D_k respectively, while the greatest common divisor of D_ν and D_k is D_δ, where δ is the greatest common divisor of ν and k. If we suppose that δ is different from ν we shall be led to a contradiction; for, F_ν is then a factor of D_ν/D_δ, as we see from (5), whereas from Theorem X it follows that D_ν/D_δ is not divisible by p since p is a factor of D_δ and ν/δ is prime to p. Hence $\delta = \nu$; and therefore k is a multiple of ν.

We shall now show that $F_{\nu p^a}(\alpha, \beta)$, $a > 0$, is divisible by p but not by p^2, except that when $p = 2$, $\nu = 3$, F_6 may be divisible by 2^2. [From Theorem III it follows that F_6 is divisible by 2.] If we suppose that we do not have simultaneously $p = 2$, $\nu = 3$, $a = 1$, we may proceed as follows: From

* When no confusion can arise we sometimes write F_ν for $F_\nu(\alpha, \beta)$.

Theorem IV we have

$$\frac{D_{\nu p^a}}{D_{\nu p^{a-1}}} = \prod_i F_i(\alpha, \beta),$$

where i ranges over those divisors of νp^a which contain the factor p^a. From Theorem X it follows that the first member of this equation is divisible by p but not by p^2. Hence (only) one of the numbers $F_i(\alpha, \beta)$ of the second member is divisible by p and it is not divisible by p^2. Suppose that this number is that for which $i = k$. Then k is a multiple of p^a. But from the discussion in the preceding paragraph we see that k is a multiple of ν. Hence $k = \nu p^a$, since this is the only common multiple of ν and p^a occurring as a subscript in the second number of our equation.

From this we conclude that each of the numbers $F_{\nu p}, F_{\nu p^2}, \cdots$ contains the factor p but that no one of them contains p^2, except that when $p = 2$, $\nu = 3$, F_6 may contain 2^2.

Now consider the number $F_{\nu\mu p^a}$, where μ is greater than unity and is prime to p. It is a divisor of $D_{\nu\mu p^a}/D_{\nu p^a}$; and from X it follows that the latter number is not divisible by p. Hence $F_{\nu\mu p^a}$ is prime to p.

Let us suppose that $F_1^2, \equiv (\alpha - \beta)^2$, is divisible by the odd prime p. From the remark following Lemma II we see that each of the numbers $F_p, F_{p^2}, \cdots$ is divisible by p. Just as in the preceding argument we may show that no one of the numbers $F_{p^2}, F_{p^3}, \cdots$ is divisible by p^2, and that $F_{\mu p^a}$ is not divisible by p if μ is greater than 1 and is prime to p and $a > 0$. The example

$$\alpha = 1 + \sqrt{6}, \quad \beta = 1 - \sqrt{6}, \quad (\alpha - \beta)^2 = 24, \quad F_3 = \alpha^2 + \alpha\beta + \beta^2 = 9$$

shows that F_1^2 may be divisible by p while F_p is at the same time divisible by p^2. If μ is greater than 1 and is prime to p and if further F_μ is divisible by p, we see at once that D_μ and D_p are both divisible by p — contrary to the corollary to Theorem VI, which asserts that D_μ and D_p are relatively prime since μ and p are relatively prime. Hence F_μ is not divisible by p.

Now suppose that F_1^2 is divisible by 2. Then, since

$$F_1^2 = (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

it follows that $\alpha + \beta$ is divisible by 2. That is, F_2 is divisible by 2. The example $\alpha = 2^k + 1$, $\beta = 2^k - 1$ shows that F_2 may be divisible by any power of 2 whatever. By means of the relation

$$F_{2^a} = \alpha^{2^{a-1}} + \beta^{2^{a-1}} = (\alpha^{2^{a-2}} + \beta^{2^{a-2}})^2 - 2\alpha^{2^{a-2}}\beta^{2^{a-2}}$$

it may be proved, however, that F_{2^a}, $a > 1$, is divisible by 2 but not by 2^2.

To be continued.

ON THE NUMERICAL FACTORS OF THE ARITHMETIC FORMS $\alpha^n \pm \beta^n$.*

By R. D. Carmichael.

Continued from page 48.

We shall now show that if $F_{\nu p^a}$ is divisible by p, where ν is prime to p, then $F_\nu{}^2$ is divisible by p. If $\nu = 1$ the result is already contained in the remark following Lemma II. From equation (10) we have

$$F_{\nu p^a}(\alpha, \beta) \cdot F_\nu(\alpha^{p^{a-1}}, \beta^{p^{a-1}}) = F_\nu(\alpha^{p^a}, \beta^{p^a}).$$

If $\nu > 1$ Lemma I is applicable, and we have

$$F_{\nu p^a}(\alpha, \beta) \cdot F_\nu(\alpha, \beta) \equiv F_\nu(\alpha, \beta) \bmod p.$$

Hence if

$$F_\nu(\alpha, \beta) \not\equiv 0 \bmod p,$$

then

$$F_{\nu p^a}(\alpha, \beta) \equiv 1 \bmod p,$$

contrary to the hypothesis that

$$F_{\nu p^a}(\alpha, \beta) \equiv 0 \bmod p.$$

From this argument we see also that when $\nu > 1$ and $F_{\nu p^a}$ is not divisible by p, then

$$F_{\nu p^a} \equiv 1 \bmod p.$$

If $\nu = 1$, we have

$$F_{p^a} \equiv (\alpha - \beta)^{(p-1)} \bmod p$$

if p is odd, as may readily be shown by means of the remark following lemma II. Evidently,

$$F_{2^a} \equiv 1 \bmod 2$$

when

$$F_{2^a} \not\equiv 0 \bmod 2.$$

Combining the several results obtained above we have the following fundamental theorem:

Theorem XIV. *Let ν be any positive integer and let p be any prime not dividing ν; then*

(1) *If $F_{\nu p^a} \equiv 0 \bmod p$, then $F_\nu{}^2 \equiv 0 \bmod p$.*

(2) *If $F_\nu{}^2 \equiv 0 \bmod p$, then each of the numbers $F_{\nu p}, F_{\nu p^2}, F_{\nu p^3}, \cdots$ is divisible by p, and none of them is divisible by p^2 except when $\nu = 1$ in which*

case F_p *may be divisible by a power of* p* *and when* $\nu = 3$, $p = 2$ *in which case* F_6 *may be divisible by* 2^2. *Moreover,* $F_{k^2} \not\equiv 0 \bmod p$ *unless* k *is of the form* νp^a.

(3) *If* $F_{\nu p^a} \not\equiv 0 \bmod p$, $a > 0$, *then* $F_{\nu p^a} \equiv 1 \bmod p$ *when* $\nu > 1$ *or when* $\nu = 1$ *and* $p = 2$; *if* $\nu = 1$ *and* p *is odd we have*

$$F_{p^a} \equiv (\alpha - \beta)^{(p-1)} \equiv \pm 1 \bmod p.$$

Let m and n be any two positive integers different from each other and from unity. We enquire what is the greatest common divisor of $F_m(\alpha, \beta)$ and $F_n(\alpha, \beta)$. If p is a prime divisor of each of these numbers it follows from the preceding theorem that there exists a number ν prime to p such that

$$F_{\nu^2} \equiv 0 \bmod p, \qquad m = \nu p^a, \qquad n = \nu p^b;$$

whence we conclude further that F_m and F_n contain p but not p^2 as a common factor. If a second prime number q is a factor of both numbers we have $m = \mu q^c$ and $n = \mu q^d$ where μ is prime to q; whence

$$\nu p^a = \mu q^c \quad \text{and} \quad \nu p^b = \mu q^d; \quad \text{or} \quad \frac{\mu}{\nu} = \frac{p^a}{q^c} = \frac{p^b}{q^d}.$$

Therefore $a = b$ and $c = d$; that is, $m = n$, contrary to the hypothesis. From these considerations we have readily the following theorem:

THEOREM XV. *If* m *and* n *are positive integers different from each other and from unity, then the greatest common divisor of* $F_m(\alpha, \beta)$ *and* $F_n(\alpha, \beta)$ *is unity or a prime* p: *a necessary and sufficient condition that it is* p *is that there exists a* ν *such that*

$$F_{\nu^2} \equiv 0 \bmod p, \quad m = \nu p^a \quad \textit{and} \quad n = \nu p^b.$$

If p is any prime number which does not divide $\alpha\beta$ then p is a factor of one of the numbers D_{p-1}, D_p, D_{p+1}, as we saw in the preceding section. From Theorem IV it follows then that p is a factor of one number at least of the set $F_2, F_3, \cdots, F_{p+1}$. Now if $F_{\nu p^a}$ is divisible by p so is F_{ν^2}, ν being prime to p; moreover, F_{k^2} is not divisible by p if k is less than ν, as we see from Theorem XIV. Hence ν is not greater than $p + 1$. In particular ν does not contain a prime factor greater than p if p is odd. We may apply this result to the problem of finding the greatest common divisor of m and $F_m(\alpha, \beta)$. We see at once that the greatest common odd divisor of these numbers is 1 or p, where p is the greatest odd prime factor of m; and that if this divisor is p and $m = \nu p^a$ where ν is prime to p, then ν is not greater than $p + 1$. There are two cases in which m and F_m may have the factor

* But if α and β are integers and p is odd it is easy to show that F_p is not divisible by p^2.

2 in common. If F_2 is divisible by 2 and m is a power of 2, then m and F_m have the greatest common divisor 2. If F_3 is divisible by 2 and m is of the form 3.2^t, then m and F_m have the factor 2 in common but not the factor 2^2. If $t = 1$ or 2 and F_{2^t} is divisible by 3 they contain also the common factor 3. These results may be put into the following theorem:

THEOREM XVI. *The greatest common odd divisor of m and $F_m(\alpha, \beta)$ is 1 or p, where p is the greatest odd prime factor of m; if this divisor is p and $m = \nu p^a$, where ν is prime to p, then ν is not greater than $p + 1$. These numbers contain in addition the common factor 2 (but not 2^2) in two cases: (a) when F_2 is divisible by 2 and m is a power of 2; (b) when F_3 is divisible by 2 and m is of the form 3.2^t.*

Now, when m is greater than 1,

$$D_m = \Pi F_d(\alpha, \beta),$$

where d runs over the divisors of m except unity. The number F_{mp}, where p is prime, has no factor in common with any F_d other than a common factor of d and mp. Hence every common prime factor of D_m and F_{mp} is likewise a factor of mp. Suppose that D_m and F_{mp} have a common prime factor q different from p. Then some F_d contains the factor q; and hence q is a divisor of some $F_\nu{}^2$ where ν is prime to q, as we see from Theorem XIV; and therefore (Theorem XIV) q is not a divisor of F_{mp}. Hence D_m and F_{mp} contain no common prime factor other than p. Again applying Theorem XIV we see that the greatest common divisor of D_m and F_{mp} is 1 or p. Hence we have the following theorem:

THEOREM XVII. *If p is a prime number then the greatest common divisor of D_m and F_{mp} is 1 or p.*

COROLLARY. *The greatest common divisor of D_m and D_{mp}/D_m is 1 or p.*

For, we have

$$\frac{D_{mp}}{D_m} = \Pi F_{kp},$$

where kp runs over those divisors of mp which are not at the same time divisors of m. Now the greatest common divisor of F_{kp} and D_m is a divisor of D_k, since k is obviously the greatest common divisor of m and kp. But the greatest common divisor of D_k and F_{kp} is 1 or p. Also, not more than one of the numbers F_{kp} contains the factor p, as one sees from Theorem XIV, since all the subscripts kp contain p to the same power. Hence the corollary.

We shall now consider a different kind of property of $F_n(\alpha, \beta)$. Denote by P_n the greatest factor of F_n which is prime to n; and write

$$F_n = \lambda P_n.$$

From Theorems XIV and XVI we see that $|\lambda|$ is unity or is the greatest

prime factor of n except possibly when n is of the form 3.2^t in which case it is a power of 2 or such a power multiplied by 3 or when $n = p$, a prime number, in which case $|\lambda|$ may be a power of p. The question arises as to the possibility that P_n shall be equal to 1; and to this we turn attention.

There are two cases in which the results are of essentially different character: (1) when $|\alpha| = |\beta|$, in which case it is obvious that α and β are complex quantities; (2) when $|\alpha| \neq |\beta|$, in which case α and β are real. In the former case P_n may be equal to unity for various values of n, as the following examples show:

(1) When

$$\alpha + \beta = 1, \qquad \alpha\beta = 2,$$

we have

$$D_{n+2} = D_{n+1} - 2D_n;$$

and therefore we may readily determine the following values:

$$n = 1, 2, \quad 3, \quad 4, \quad 5, 6, 7, \quad 8, \quad 9, \quad 10, 11, 12, \quad 13, \quad 14, \quad 15, \quad 16, \quad 17, 18,$$

$$D_n = 1, 1, -1, -3, -1, 5, 7, -3, -17, -11, 23, 45, -1, -91, -89, 93, 271, 85,$$

$$P_n = 1, 1, \quad 1, \quad 3, \quad 1, 5, 1, \quad 1, \quad 17, \quad 11, 23, \quad 1, \quad 1, \quad 13, \quad 89, \quad 31, 271, \quad 1.$$

Here $P_n = 1$ for $n = 1, 2, 3, 5, 7, 8, 12, 13, 18$.

(2) When $\alpha + \beta = 5$, $\alpha\beta = 7$, we have

$$F_{10} = \frac{\alpha^5 + \beta^5}{\alpha + \beta} = -5.$$

Hence in this case $P_{10} = 1$.

(3) If $\alpha + \beta = 2$, $\alpha\beta = 7$, then $F_8 = \alpha^4 + \beta^4 = 2$. Hence $P_8 = 1$.

(4) If $\alpha + \beta = 3$, $\alpha\beta = 4$, then $P_4 = 1$, $P_6 = 1$.

These examples show that for appropriate values of α and β $P_n(\alpha, \beta)$ may be unity for various values of n; in particular for all values less than 14 except possibly 9 and 11. It is to be noted, however, that in all these examples $|\alpha| = |\beta|$. If α and β are real, and hence $|\alpha| \neq |\beta|$, we have a different state of affairs as will now be proved. The treatment will fall into two parts, according as α and β are of the same or of different sign.

Let us suppose first that the real quantities α and β are of the same sign. Without loss of generality we may take them to be positive; and this we do. Further, since

$$F_n(\alpha, \beta) = F_n(\beta, \alpha),$$

there is no loss of generality in assuming, as we shall do, that α is greater

than β. Now

$$F_2(\alpha, \beta) = \alpha + \beta$$

and this number is to a large extent arbitrary; hence we shall suppose that n is greater than 2. Moreover, there is an infinite number of cases in which $P_6 = 1$, as we see by putting

$$\alpha + \beta = 3 + 2^k, \qquad \alpha\beta = 3 + 2^{k+1},$$

whence

$$F_6 = \alpha^2 - \alpha\beta + \beta^2 = (\alpha + \beta)^2 - 3\alpha\beta = 2^{2k}.$$

Therefore we leave out of consideration entirely the case when $n = 6$.

From equation (9) we have

$$(20) \qquad F_n(\alpha, \beta) = \lambda P_n(\alpha, \beta) = \frac{D_n \cdot \Pi D_{n/p_ip_j} \cdots}{\Pi D_{n/p_i} \cdot \Pi D_{n/p_ip_jp_k} \cdots},$$

where the products denoted by Π extend over the combinations 2, 4, 6, $\cdots$ at a time of p_1, p_2, $\cdots$ in the numerator and over the combinations 1, 3, 5, $\cdots$ at a time in the denominator, p_1, p_2, $\cdots$ being the different prime factors of n.

Now we have

$$D_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} = \alpha^{k-1} + \alpha^{k-2}\beta + \cdots + \beta^{k-1} > \alpha^{k-1}.$$

Applying this inequality to the numerator of the last member of (20) we have

$$D_n \cdot \Pi D_{n/p_ip_j} \cdots > \alpha^\sigma,$$

where

$$\sigma = (n - 1) + \Sigma\left(\frac{n}{p_ip_j} - 1\right) + \Sigma\left(\frac{n}{p_ip_jp_kp_l} - 1\right) + \cdots$$

$$= n - 2^{m-1} + \Sigma\frac{n}{p_ip_j} + \Sigma\frac{n}{p_ip_jp_kp_l} + \cdots,$$

where m is the number of different prime factors of n.

Now

$$D_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} < \frac{\alpha^k}{\alpha - \beta} < \alpha^k,$$

since $(\alpha - \beta)^2$ is an integer and $\alpha - \beta$ is positive. Applying this inequality to the denominator of the last member of (20) we have

$$\Pi D_{n/p_i} \cdot \Pi D_{n/p_ip_jp_k} \cdots < \alpha^\tau,$$

where

$$\tau = \Sigma\frac{n}{p_i} + \Sigma\frac{n}{p_ip_jp_k} + \cdots.$$

By division we see that the last member of (20) is greater than $\alpha^{\sigma-\tau}$. Now

$$\sigma - \tau = -2^{m-1} + n - \Sigma\frac{n}{p_i} + \Sigma\frac{n}{p_ip_j} - \Sigma\frac{n}{p_ip_jp_k} + \cdots$$
$$= \varphi(n) - 2^{m-1},$$

where $\varphi(n)$ is Euler's φ-function of n, since

$$\varphi(n) = n - \Sigma\frac{n}{p_i} + \Sigma\frac{n}{p_ip_j} - \cdots.$$

Therefore

(22) $$F_n(\alpha, \beta) = \lambda P_n(\alpha, \beta) > \alpha^{\phi(n)-2^{m-1}}.$$

If we apply the inequality $D_k < \alpha^k$ to the numerator of (20) and the inequality $D_k > \alpha^{k-1}$ to its denominator we find that

(23) $$F_n(\alpha, \beta) = \lambda P_n(\alpha, \beta) < \alpha^{\phi(n)+2^{m-1}}.$$

Now

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta.$$

Since $\alpha - \beta$ is real and different from zero we have $(\alpha + \beta)^2 > 4\alpha\beta$. But $\alpha\beta$ is a positive integer, according to our present hypothesis; and hence

$$\alpha + \beta \gtreqq 3.$$

Also,

$$\alpha - \beta \gtreqq 1.$$

Hence $\alpha \gtreqq 2$. This fact is of use in connection with relations (22) and (23).

It is obvious that

$$\varphi(n) \gtreqq 2^{m-1}.$$

Hence from relation (22) we see that if $P_n = 1$ then $\lambda > 1$. Hence if n is neither an odd prime nor of the form 3.2^t, $t > 1$, we must have λ equal to the greatest prime factor of n, since, as we have seen before, this is now the only possible value of λ different from unity.

Let us consider first the case when n is of the form $n = 3.2^t$. Here we have from (22),

$$\lambda P_n(\alpha, \beta) > \alpha^{2^{t-2}} \gtreqq 2^{2^{t-2}},$$

where λ has one of the values 2, 3, 6 (Theorems XIV and XVI). Hence if $t > 2$ we have $P_n > 1$. Hence we have to examine further only the case when $t = 2$. To show that $P_{12} > 1$ it is sufficient to show that $F_{12} > 6$. Since $(\alpha + \beta)^2 - 4\alpha\beta > 0$ we have $\alpha^2 + \beta^2 > 2\alpha\beta$; and therefore

$$F_{12} \equiv \alpha^4 - \alpha^2\beta^2 + \beta^4 = (\alpha^2 + \beta^2)^2 - 3\alpha^2\beta^2 > \alpha^2\beta^2.$$

Hence $F_{12} > 6$ unless $\alpha\beta \leqq 2$. Considering this case we see that

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \gtreqq 9 - 4 = 5,$$

since

$$\alpha + \beta \gtreqless 3.$$

Hence

$$F_{12} \gtreqless 5^2 - 12 > 6.$$

Hence $P_{12} > 1$ for every positive α and β.

We consider next the case when λ is equal to the greatest prime factor of n. Call this factor p and write $n = \nu p^a$, where ν is prime to p. Suppose first that ν is not unity (whence $p > 2$). Applying relation (23) we have

$$(23') \qquad F_\nu(\alpha, \beta) < \alpha^{\phi(\nu)+2^{m-2}}.$$

Now

$$F_\nu(\alpha, \beta) \equiv 0 \bmod p,$$

as we see from Theorem XIV, since in the present case p is a factor of $F_n(\alpha, \beta)$. But $F_\nu(\alpha, \beta)$ is different from zero; and hence it is equal to or greater than p. Therefore

$$P_n(\alpha, \beta) = \frac{F_n(\alpha, \beta)}{p} \gtreqless \frac{F_n(\alpha, \beta)}{F_\nu(\alpha, \beta)} > \alpha^{\phi(n)-\phi(\nu)-3\cdot 2^{m-2}};$$

the last term in this relation is obtained by aid of (22) and (23′). Hence $P_n > 1$ except when $\varphi(n) - \varphi(\nu) - 3.2^{m-2}$ is negative. Now

$$\varphi(n) = p^{a-1}(p-1)\varphi(\nu);$$

and therefore the exponent above may be written in the form

$$\varphi(\nu)[p^{a-1}(p-1) - 1] - 3.2^{m-2}.$$

Since

$$\varphi(\nu) \gtreqless 2^{m-2}$$

the expression above can be negative only when $p^{a-1}(p-1) - 1 < 3$. Since $p \neq 2$, this condition can be satisfied only when $p^a = 3$. Then n is of the form 3.2^t, a case which we have already treated.

In case $\nu = 1$ we have from (22)

$$pP_{p^a} > 2^{p^{a-1}(p-1)-1} \gtreqless 2^{p-2}.$$

If we now assume that $P = 1$ it follows from the last term of these inequalities that $p = 2$ or 3, since $2^{p-2} > p$ when $p \gtreqless 5$. Then from the middle term we see that a must be unity. Therefore, since we are leaving out of consideration the case when $n = 2$, we have to examine further only the case when

$$n = p^a = 3.$$

Now

$$F_3(\alpha, \beta) = \alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta > 3\alpha\beta,$$

since $(\alpha + \beta)^2 > 4\alpha\beta$. Hence $F_3 > 3$, so that in this case $P_3 > 1$.

There yet remains the case when n is equal to an odd prime p and $\lambda = p^s$, where s is an integer greater than unity. We have seen (footnote to Theorem XIV) that in this case α and β are not integers. If $P_p = 1$ we have $F_p = p^s$. Now from Lemma II in § 3 and the fact that $D_p = F_p$ we have

$$F_p \equiv (\alpha - \beta)^{p-1} \bmod p.$$

And hence the condition $F_p = p^s$ cannot be satisfied unless p is a factor of $(\alpha - \beta)^2$. As an example to show that this possibility can actually arise we have

$$\alpha + \beta = 3^k + 1, \qquad \alpha\beta = \tfrac{1}{4}\{(3^k + 1)^2 - 3(3^k - 1)^2\},$$

$$F_3(\alpha, \beta) = (\alpha + \beta)^2 - \alpha\beta = 3^{k+1},$$

where k is any positive integer.

Thus we are led to the following preliminary result:

If α and β are real and of the same sign and $n \neq 1, 2, 6$, then $F_n(\alpha, \beta)$ contains a factor (other than unity) which is prime to n except when α and β are suitably chosen irrational numbers and, at the same time, n is equal to an odd prime factor of $(\alpha - \beta)^2$.

COROLLARY.* *If a and b are relatively prime positive integers, $a > b$, and $n > 2$, then $F_n(a, b)$ contains a factor (other than unity) which is prime to n except when $n = 6$, $a = 2$, $b = 1$.* (Compare the more general result in Theorem XIX.)

To complete the proof of the corollary it is sufficient to show that $a = 2$ and $b = 1$ are the only positive integer values of a and b, $a > b$, for which

$$F_6 \equiv a^2 - ab + b^2 = 2^k \cdot 3^\rho,$$

k and ρ being integers; and this is easily done.

Let us turn now to the case when α and β are real but of different sign. As before, we exclude from consideration the cases $n = 1, 2, 6$. Since

$$P_n(\beta, \alpha) = P_n(\alpha, \beta) = P_n(-\alpha, -\beta)$$

we may without loss of generality assume that α is positive and greater than $|\beta|$; and this we do. Now

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta \geqq 5.$$

Hence

$$\alpha - \beta \geqq \sqrt{5} > 2.$$

Also,

$$\alpha + \beta \geqq 1.$$

Hence $\alpha > 3/2$. These inequalities will be useful in the sequel.

* The theorem of the corollary is due to Birkhoff and Vandiver, l. c., pp. 177–179. Compare their method of treatment with that in the text.

We have

$$D_{2k} = \frac{\alpha^{2k} - \beta^{2k}}{\alpha^2 - \beta^2}(\alpha + \beta) = (\alpha + \beta)(\alpha^{2k-2} + \alpha^{2k-4}\beta^2 + \cdots + \beta^{2k-2}) > \alpha^{2k-2};$$

$$D_{2k} < \frac{\alpha^{2k}}{\alpha - \beta} < \frac{\alpha^{2k}}{2};$$

$$D_{2k+1} = \frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta} > \frac{\alpha^{2k+1}}{2\alpha} = \frac{\alpha^{2k}}{2} > \alpha^{2k-2};$$

$$D_{2k+1} < \frac{2\alpha^{2k+1}}{\alpha} = 2\alpha^{2k} < \alpha^{2k+2}.$$

1. Now suppose that n is a multiple of 4. Applying the inequality $D_{2k} < \frac{1}{2}\alpha^{2k}$ to the denominator of (20) we have

$$\Pi D_{n/p_i} \cdot \Pi D_{n/p_ip_jp_k} \cdots < 2^{-2^{m-1}}\alpha^{\tau},$$

where

$$\tau = \Sigma\frac{n}{p_i} + \Sigma\frac{n}{p_ip_jp_k} + \cdots,$$

m being as before the number of different prime factors of n. Likewise by means of the inequality $D_{2k} > \alpha^{2k-2}$ we see that

$$D_n \cdot \Pi D_{n/p_ip_j} \cdots > \alpha^{\sigma},$$

where

$$\sigma = -2^m + n + \Sigma\frac{n}{p_ip_j} + \cdots.$$

Hence by division in the last member of (20) and further reduction we have

$$F_n(\alpha, \beta) = \lambda P_n(\alpha, \beta) > 2^{2^{m-1}}\alpha^{\phi(n)-2^m} > \alpha^{\phi(n)} > 2^{\frac{1}{2}\phi(n)}, \tag{24}$$

since $\alpha^2 > 2$.

If n is a power of 2 then λ is 1 or 2 (Theorem XVI); and hence we see from (24) that P_n cannot in this case be unity. Furthermore if $P_n = 1$ n can contain no odd prime factor greater than 3; for when n contains the odd prime factor p the last exponent in (24) is equal to or greater than $p - 1$ and $2^{p-1} > 2p$ unless $p = 3$, and hence greater than λ. It is easy to see in a similar manner that n cannot contain 3^2 or 2^3. Hence the only case left for special consideration is that for which $n = 12$. From (24) it follows that $F_{12} > 4$, and hence if $P_{12} = 1$ we must have $\lambda = 6$, since its only possible values (Theorems XIV and XVI) are 1, 2, 3, 6. This requires that $F_{12} = 6$, which we show to be possible when and only when

$$\alpha + \beta = 1, \qquad \alpha\beta = -1.$$

For, from (24) it follows that $F_{12} > \alpha^4$; hence $\alpha < 2$, whence $|\beta| < 2$. Therefore

$$\alpha + \beta = 1.$$

Now

$$F_{12} = \alpha^4 - \alpha^2\beta^2 + \beta^4 = (\alpha + \beta)^4 - 4\alpha\beta(\alpha + \beta)^2 + \alpha^2\beta^2.$$

Hence

$$F_{12} = 1 - 4\alpha\beta + \alpha^2\beta^2 = 6;$$

and therefore

$$\alpha\beta = -1.$$

2. We take next the case when n is odd. If we apply to the denominator in (20) the inequality $D_{2k+1} < 2\alpha^{2k}$ and to the numerator the inequality $D_{2k+1} > \frac{1}{2}\alpha^{2k}$, we obtain the relation

$$F_n(\alpha, \beta) = \lambda P_n(\alpha, \beta) > 2^{-2^m}\alpha^{\phi(n)} > 2^{\frac{1}{2}\phi(n)-2^m}. \tag{25}$$

Now λ is unity or the greatest odd prime factor of n.* Assuming $P_n \neq 1$ it is easy to see from the above inequalities that n does not contain as many as three different prime factors; that if it contains two prime factors one of these must be 3 and the other 5 or 7, while no one of them is repeated; that if it contains only a single prime factor this prime is not greater than 11 and occurs only to the first power unless it is 3, when it may occur to the second power. Thus we have to examine further only the cases $n = 3$, 3^2, 5, 7, 11, 15, 21. Furthermore we see from (25) that λ cannot be unity except possibly in the case $n = 3$, so that if $P_n = 1$ we have F_n equal to the greatest prime factor of n except possibly when $n = 3$.

Now

$$F_3 = \alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta \geqq 2.$$

Hence if $P_3 = 1$, $\lambda = 3$ and $F_3 = 3$,—an equation which can be satisfied when and only when

$$\alpha + \beta = 1, \qquad \alpha\beta = -2.$$

Thus we have a single exceptional case when $P_3 = 1$.

If $P_9 = 1$ we must have

$$F_9 = \alpha^6 + \alpha^3\beta^3 + \beta^6 = (\alpha^3 + \beta^3)^2 - \alpha^3\beta^3 = 3,$$

which is clearly impossible. If $P_5 = 1$ we have

$$F_5 = \alpha^4 + \alpha^3\beta + \cdots + \beta^4 = (\alpha + \beta)^4 - 3\alpha\beta(\alpha + \beta)^2 + \alpha^2\beta^2 = 5.$$

Since $\alpha\beta$ is negative it follows readily that this equation can be satisfied when and only when

$$\alpha + \beta = 1, \qquad \alpha\beta = -1.$$

* In the text we are tacitly laying aside the case when λ is a power (greater than the first) of the greatest prime factor p of n; for this case, as we have seen, cannot arise unless p is a factor of $(\alpha - \beta)^2$ and $n = p$.

Thus we have here an exceptional case.

If $P_7 = 1$ we have

$$F_7 = \alpha^6 + \alpha^5\beta + \cdots + \beta^6 = (\alpha+\beta)^6 - 5\alpha\beta(\alpha+\beta)^4 + 6\alpha^2\beta^2(\alpha+\beta)^2 - \alpha^3\beta^3 = 7,$$

which is obviously impossible when $\alpha\beta$ is negative. We have

$$F_{11} = D_{11} > \tfrac{1}{2}\alpha^{10} > 2^4 > 11,$$

and therefore $P_{11} \neq 1$. Also,

$$F_{15} = \alpha^8 - \alpha^7\beta + \alpha^5\beta^3 - \alpha^4\beta^4 + \alpha^3\beta^5 - \alpha\beta^7 + \beta^8$$
$$= (\alpha+\beta)^8 - 9\alpha\beta(\alpha+\beta)^6 + 26\alpha^2\beta^2(\alpha+\beta)^4 - 24\alpha^3\beta^3(\alpha+\beta)^2 + \alpha^4\beta^4.$$

Hence $F_{15} > 5$ when $\alpha\beta$ is negative; and therefore $P_{15} \neq 1$. We have

$$F_{21} = \frac{D_{21}}{D_3 \cdot D_7} > \frac{\frac{1}{2}\alpha^{20}}{2\alpha^2 \cdot 2\alpha^6} = \frac{\alpha^{12}}{2^3} > 2^3 > 7;$$

and hence $P_{21} \neq 1$.

Thus we have completed the investigation of the case when n is odd.

3. Let us consider finally the case in which n is an odd multiple of 2. From the inequalities $D_{2k} > \alpha^{2k-2}$ and $D_{2k+1} > \alpha^{2k-2}$ we see that in general $D_\nu > \alpha^{\nu-3}$; and from the inequalities $D_{2k} < \frac{1}{2}\alpha^{2k}$ and $D_{2k+1} < \alpha^{2k+2}$ we see that $D_\nu < \alpha^{\nu+1}$. Applying the inequalities $D_\nu > \alpha^{\nu-3}$ and $D_\nu < \alpha^{\nu+1}$ to the numerator and the denominator respectively of (20) we obtain without difficulty

(26) $$F_n(\alpha, \beta) = \lambda P_n(\alpha, \beta) > \alpha^{\phi(n)-2^{m+1}}.$$

Now λ is either unity or the greatest prime factor of n. Then, since $\alpha > 3/2$ we see from (26) that if $P_n = 1$ n contains no prime factor greater than 13 and that such prime factor can enter only to the first power except in the case of 3 which may enter to the second power; that if n contains two odd prime factors one of these is 3 and the other is 5 or 7 or 11 while no one of them is repeated; and that n does not contain three different odd prime factors.

Let $n = 2 \cdot 3 \cdot p$ where p is 5, 7 or 11. Then

$$F_{6p} = \frac{D_{6p} \cdot D_p \cdot D_3 \cdot D_2}{D_{3p} \cdot D_{2p} \cdot D_6 \cdot D_1} > \frac{\alpha^{6p-2} \cdot \frac{1}{2}\alpha^{p-1} \cdot \frac{1}{2}\alpha^2 \cdot 1}{2\alpha^{3p-1} \cdot \frac{1}{2}\alpha^{2p} \cdot \frac{1}{2}\alpha^6 \cdot 1} = \tfrac{1}{2}\alpha^{2p-6} > 2^{p-4}.$$

From this inequality it follows that the only possible value for p is 5. For this case we have

$$F_{30} = \alpha^8 + \alpha^7\beta - \alpha^5\beta^3 - \alpha^4\beta^4 - \alpha^3\beta^5 + \alpha\beta^7 + \beta^8$$
$$= (\alpha+\beta)^8 - 7\alpha\beta(\alpha+\beta)^6 + 14\alpha^2\beta^2(\alpha+\beta)^4 - 8\alpha^3\beta^3(\alpha+\beta)^2 + \alpha^4\beta^4 > 5.$$

Hence $P_{30} \neq 1$.

Suppose next that $n = 2p$, where p is 5, 7, 11 or 13. Then

$$F_{2p} = \frac{D_{2p} \cdot D_1}{D_p \cdot D_2} > \frac{\alpha^{2p-2}}{2\alpha^{p-1} \cdot \frac{1}{2}\alpha^2} = \alpha^{p-3} > \left(\frac{3}{2}\right)^{p-3}.$$

From these relations it follows that $p = 5$ or 7. For $p = 5$ we have

$$F_{10} = \alpha^4 - \alpha^3\beta + \alpha^2\beta^2 - \alpha\beta^3 + \beta^4 = (\alpha + \beta)^4 - 5\alpha\beta(\alpha + \beta)^2 + 5\alpha^2\beta^2 > 5;$$

and hence $P_{10} \neq 1$. For $p = 7$ we have

$$\begin{aligned} F_{14} &= \alpha^6 - \alpha^5\beta + \alpha^4\beta^2 - \alpha^3\beta^3 + \alpha^2\beta^4 - \alpha\beta^5 + \beta^6 \\ &= (\alpha + \beta)^6 - 7\alpha\beta(\alpha + \beta)^4 + 14\alpha^2\beta^2(\alpha + \beta)^2 - 7\alpha^3\beta^3 > 7; \end{aligned}$$

and hence $P_{14} \neq 1$.

There remains yet the case $n = 18$. We have

$$F_{18} = \alpha^6 - \alpha^3\beta^3 + \beta^6 = (\alpha^3 + \beta^3)^2 - 3\alpha^3\beta^3 > 3.$$

Hence $P_{18} \neq 1$.

The various results contained in the immediately preceding discussion may be summarized into the following theorem:

THEOREM XVIII. *If α and β are real and if $n \neq 1, 2, 6$, then $F_n(\alpha, \beta)$ contains a factor (other than unity) which is prime to n in all cases except when: (a) α and β are suitably chosen irrational numbers and n is equal to an odd prime factor of $(\alpha - \beta)^2$;*

(*b*)	$n = 3,$	$\alpha + \beta = \pm 1,$	$\alpha\beta = -2;$
(*c*)	$n = 5,$	$\alpha + \beta = \pm 1,$	$\alpha\beta = -1;$
(*d*)	$n = 12,$	$\alpha + \beta = \pm 1,$	$\alpha\beta = -1.$

The following particular case is of sufficient importance to merit separate statement:

THEOREM XIX. *If a and b are any relatively prime integers and $n > 2$, then $F_n(a, b)$ contains a factor (other than unity) which is prime to n in all cases except when*

(*a*)	$n = 3,$	$a + b = \pm 1,$	$ab = -2;$
(*b*)	$n = 6,$	$a + b = \pm 3,$	$ab = 2.$

To complete the proof of this theorem it is further necessary (and sufficient) to determine the general solution in relatively prime integers $a + b$ and ab of the equation

$$F_6(a, b) = a^2 - ab + b^2 = (a + b)^2 - 3ab = 2^k \cdot 3^\rho,$$

where k and ρ are integers. This is easily done, and the work is omitted here.

Lucas (l. c., p. 199) makes the statement that neither D_n nor S_n can be

prime unless n is prime. The treatment in this section makes it easy to construct examples to show that this statement is not accurate. Thus if $\alpha + \beta = 1$ and $\alpha\beta = 2$, we have

$$D_4 = -3, \quad D_6 = 5, \quad D_8 = -3, \quad D_9 = -17, \quad D_{10} = -11,$$
$$D_{15} = -89, \quad D_{26} = 181; \quad S_9 = -5.$$

Equations (5) and (11) make it evident, however, that a prime value for D_n or S_n, when n is not prime, is of relatively rare occurrence.

5. Characteristic Factors of F_n, D_n, S_n.

The number $F_1 = \alpha - \beta$ is not in general an integer, but F_1^2 is always an integer. Consider then the sequence of integers $F_1^2, F_2, F_3, \cdots$. By a characteristic factor of F_n we mean a prime divisor of F_n which is not a factor of any number of the set $F_1^2, F_2, \cdots, F_{n-1}$.

Similarly, a characteristic factor of $D_n(S_n)$ is a prime divisor of $D_n(S_n)$ which is not a factor of any $D_\nu(S_\nu)$ for which ν is less than n.

Examples given in the preceding section show that when α and β are complex quantities it may often happen that F_n, D_n and S_n do not possess characteristic factors. Hence in the " existence theorems " of the present section we confine attention to the case when α and β are real.

From Theorem XIV we have the following result:

THEOREM XX. *A necessary and sufficient condition that a prime p which divides F_n shall be a characteristic factor of F_n is that p shall not be a divisor of n.*

From Theorems XVIII and XX it is easy to deduce the following:

THEOREM XXI. *If α and β are real and if $n \neq 1, 2, 6$, then $F_n(\alpha, \beta)$ contains at least one characteristic factor in all cases except when: (a) α and β are suitably chosen irrational numbers and n is equal to an odd prime divisor of $(\alpha - \beta)^2$;*

(b) $\qquad n = 3, \quad \alpha + \beta = \pm 1, \quad \alpha\beta = -2;$

(c) $\qquad n = 5, \quad \alpha + \beta = \pm 1, \quad \alpha\beta = -1;$

(d) $\qquad n = 12, \quad \alpha + \beta = \pm 1, \quad \alpha\beta = -1.$

From equations (5) and (11) follows the theorem:

THEOREM XXII. *A characteristic factor of $F_n(F_{2n})$ is also a characteristic factor of $D_n(S_n)$.*

Hence,*

THEOREM XXIII. *If α and β are real and $n \neq 1, 2, 6$, then D_n contains*

* Compare Birkhoff and Vandiver, l. c., p. 177, and Lucas, l. c., p. 291. For the case when α and β are integers Lucas states that for n sufficiently large it is "evident" that D_n has one or more characteristic factors. In what way the fact is "evident" is not clear.

at least one characteristic factor except when

$$n = 12, \quad \alpha + \beta = \pm 1, \quad \alpha\beta = -1.$$

THEOREM XXIV. *If α and β are real and $n \neq 1, 3$, then S_n contains at least one characteristic factor except when*

$$n = 6, \quad \alpha + \beta = \pm 1, \quad \alpha\beta = -1.$$

These results, together with Theorem XIX, lead to the following:

THEOREM XXV. *If a and b are relatively prime integers and $n > 2$, then S_n contains at least one characteristic factor except when*

$$n = 3, \quad a + b = \pm 3, \quad ab = 2,$$

while D_n contains at least one characteristic factor except when

$$n = 6, \quad a + b = \pm 3, \quad ab = 2.$$

Suppose now that p is an odd characteristic factor of $F_n(\alpha, \beta)$, $n > 1$. Then p is prime to $(\alpha - \beta)^2$ by definition of characteristic factor and to $\alpha\beta$ by Theorem I in connection with equation (5). Hence from Theorem XII it follows that p is a divisor of D_{p-1} or of D_{p+1} according as $(\alpha - \beta)^{p-1}$ is congruent to $+1$ or to -1 modulo p. From this, in view of equation (5) and Theorem XIV, it follows that n is a factor of $p - \left(\frac{\alpha, \beta}{p}\right)$, where $\left(\frac{\alpha, \beta}{p}\right) = +1$ or -1 according as $(\alpha - \beta)^{p-1}$ is congruent to $+1$ or to -1 modulo p. Hence p is of the form $p = kn + \left(\frac{\alpha, \beta}{p}\right)$, and we have the following theorem:

THEOREM XXVI. *A characteristic factor of $F_n(\alpha, \beta)$ is of the form $kn + \left(\frac{\alpha, \beta}{p}\right)$.*

COROLLARY. *If a and b are relatively prime integers a characteristic factor of $F_n(a, b)$ is of the form $kn + 1$.*

This theorem and corollary lead at once to the known results (Lucas, l. c., p. 291) concerning the form of the characteristic factors of D_n and S_n.

The above theorem gives the linear form of a characteristic factor of F_n. We may also determine a quadratic form of which F_n, and consequently any one of its factors, is a divisor. We have

$$(\alpha - \beta)^2 D_n^2 = S_n^2 - 4\alpha^n\beta^n,$$

as one may readily verify. Then, since F_n is a divisor of D_n by equation (5), if we take n odd we have at once the following result:

THEOREM XXVII. *The number $F_{2k+1}(\alpha, \beta)$ is a factor of the quadratic form $x^2 - \alpha\beta y^2$.*

The equation above may also be written

$$S_n^2 = 4\alpha^n\beta^n + (\alpha - \beta)^2 D_n^2 = \frac{4\alpha^{n+1}\beta^{n+1} + \alpha\beta(\alpha - \beta)^2 D_n^2}{\alpha\beta}.$$

Then, since F_{2n} is a divisor of S_n by equation (11) and is also prime to $\alpha\beta$ by Theorem I and equation (5), we have

THEOREM XXVIII. *The numbers F_{4k+2} and F_{4k} are divisors of the quadratic forms $x^2 + \alpha\beta(\alpha - \beta)^2 y^2$ and $x^2 + (\alpha - \beta)^2 y^2$ respectively.*

As an example let us consider the case when $\alpha\beta = 3$ and $\alpha + \beta$ is not divisible by 3. Then an odd characteristic factor of F_{2k+1} is a divisor of the quadratic form $x^2 - 3y^2$; and hence is of one of the linear forms $12z \pm 1$. It is also of one of the linear forms $m(2k + 1) \pm 1$. In particular a characteristic factor of F_{25} is of one of the forms

$$300s \pm 1, \qquad 300s \pm 49.$$

Moreover, F_{25} can contain no prime factor which is not characteristic unless $(\alpha - \beta)^2$ is a multiple of 5. But

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta,$$

and is not a multiple of 5, since this would require the impossible relation

$$(\alpha + \beta)^2 \equiv 2 \bmod 5.$$

Hence, in the present case, every prime factor of F_{25} is of one of the forms given above. As a particular example, if $\alpha + \beta = 2$, we have $F_{25} = 56{,}149$. Since this number is less than 299^2 and is not divisible by 251, it is easy to see that it must be prime.

6. Some Applications of the Preceding Results.

One of the results in Theorem XIV may be used to obtain a very simple proof of the following special case of Dirichlet's celebrated theorem concerning the prime terms of an arithmetical progression of integers:

THEOREM XXIX. *In the sequence of positive integers $p^k x - 1$, where k is a positive integer, p an odd prime and x runs over the set $x = 1, 2, 3, \cdots$, there is an infinitude of prime numbers.*

To prove this theorem we choose α and β so that $(\alpha - \beta)^2$ is a quadratic non-residue of p; that is, so that

$$(\alpha - \beta)^{p-1} \equiv -1 \bmod p.$$

That this is always possible is readily seen as follows: Let ρ be any quad-

ratic non-residue of p of the form $4m + 1$. Let σ be any odd integer which is prime to ρ. Then write

$$(\alpha - \beta)^2 = \rho, \qquad \alpha + \beta = \sigma,$$

whence

$$\alpha\beta = \tfrac{1}{4}(\sigma^2 - \rho).$$

Then obviously $\alpha + \beta$ and $\alpha\beta$ are relatively prime integers; and therefore the determination of α and β from the above equations suffices for our purpose.

Now from the latter part of Theorem XIV we see that with these values of α and β we have

$$F_{p^{2r+1}} \equiv -1 \bmod p,$$

where r is a positive integer. Hence every prime factor of $F_{p^{2r+1}}$ is a characteristic factor; and any such factor, as we have seen before, is of the form $mp^{2r+1} - 1$. Consider the set of numbers

$$F_{p^{2r+1}}, \qquad r = 1, 2, 3, \cdots.$$

When $2r + 1$ is greater than k the rth number of the set contains a prime factor of the form $p^kx - 1$. But the numbers of the set are prime each to each (Theorem XV). Therefore there is an infinitude of primes of the form $p^kx - 1$, as was to be proved.

In a similar way, and even more readily, it may be shown that the sequence $nx + 1$, $x = 1, 2, 3, \cdots$, always contains an infinitude of prime numbers. For this result compare Lucas, l. c., p. 291, and Birkhoff and Vandiver, l. c., p. 177.

A large number of other special cases of Dirichlet's celebrated theorem may be simply proved by modifications of the foregoing method. This remark will be sufficiently illustrated by a proof of the following theorem:

THEOREM XXX. *There is an infinitude of prime numbers of the form* $2^k \cdot 3x - 1$, *where k is any positive integer.*

To prove this take

$$\alpha + \beta = 5, \qquad \alpha\beta = 7,$$

whence

$$(\alpha - \beta)^2 = -3.$$

Then (Theorem XXVIII) every divisor of F_{2^n}, $n > 1$, is a factor of the quadratic form $x^2 - 3y^2$. If it is an odd prime it is therefore of one of the linear forms $12z + 1$, $12z - 1$. If $n \geqq k$, it is also of one of the forms $2^kz + 1$, $2^kz - 1$. Hence it is of one of the forms $2^k \cdot 3x + 1$, $2^k \cdot 3x - 1$.

Now we have

$$F_{2^n} = \alpha^{2^{n-1}} + \beta^{2^{n-1}} = (\alpha^{2^{n-2}} + \beta^{2^{n-2}})^2 - 2\alpha^{2^{n-2}}\beta^{2^{n-2}} = F^2_{2^{n-1}} - 2\alpha^{2^{n-2}}\beta^{2^{n-2}};$$

and therefore in the present case F_{2^n} may be determined by the following method of recursion:

$$F_2 = 5, \quad F_{2^2} = 5^2 - 2 \cdot 7 = 11, \quad F_{2^3} = 11^2 - 2 \cdot 7^2 = 23, \quad \cdots.$$

It is obvious from this that every F_{2^n} is of the form $6k - 1$. Therefore, since every prime factor of F_{2^n}, $n \gtreqless k$, is of one of the forms $2^k \cdot 3x \pm 1$ it follows that for every n greater than k F_{2^n} contains at least one prime factor of the form $2^k \cdot 3x - 1$. Moreover F_{2^m} and F_{2^n} have no common factor (other than unity) when m and n are different (Theorem XV). Hence the set $F_2, F_{2^2}, F_{2^3}, \cdots$ contains as divisors an infinitude of prime numbers of the form $2^k \cdot 3x - 1$. Hence the theorem as stated above.

This theorem taken in connection with the fact that there is an infinitude of primes of the form $nx + 1$ leads to the following interesting corollary, also a special case of Dirichlet's theorem:

COROLLARY. *There is an infinitude of prime numbers of each of the forms* $4n + 1$, $4n - 1$, $6n + 1$, $6n - 1$.

7. On the Verification of Large Prime Numbers.

Lucas (l. c., pp. 301–317) has given some remarkable theorems which suffice for the determination of large prime numbers of given forms; these theorems grow out of a fundamental result which Lucas states essentially in the following form:

If D_ν is divisible by p for $\nu = p - 1$ and for no value of ν which is a factor of $p - 1$ then p is prime; likewise, if D_ν is divisible by p for $\nu = p + 1$ and for no value of ν which is a divisor of $p + 1$ then p is prime.

As thus stated the theorem is not entirely accurate; it fails when $p = 4$, as is shown by the following example:

$$\alpha + \beta = 3, \quad \alpha\beta = 1, \quad D_1 = 1, \quad D_2 = 3, \quad D_3 = 8.$$

The theorem is true, however, if the further restriction is made that p is odd. It becomes, then, a corollary of our Theorems XXXI and XXXVI below.

Some of our results concerning the numbers $F_n(\alpha, \beta)$ enable us to state general theorems which are related to those of Lucas, but are simpler in form and at the same time more far-reaching and complete. In this section we prove these theorems and develop certain of their consequences.

We begin with the following three closely related theorems:

THEOREM XXXI. *A necessary and sufficient condition that a given odd number p is prime is that there exist relatively prime integers $\alpha + \beta$ and $\alpha\beta$ such that $F_{p-1}(\alpha, \beta)$ is divisible by p.*

THEOREM XXXII. *A necessary and sufficient condition that a given odd number p is prime is that there exist relatively prime integers a and b such that $F_{p-1}(a, b)$ is divisible by p.*

THEOREM XXXIII. *A necessary and sufficient condition that a given odd number p is prime is that an integer a exists such that $F_{p-1}(a, 1)$ is divisible by p.*

For the demonstration of these *three* theorems it is obviously sufficient to prove the following two statements: If p is prime an integer a exists such that $F_{p-1}(a, 1)$ is divisible by p; if $F_{p-1}(\alpha, \beta)$ is divisible by p then p is prime. We consider separately the proofs of the two statements.

If p is an odd prime number then there exists a positive integer a (less than p) such that $a^x - 1$ is divisible by p for $x = p - 1$ but for no smaller value of x, as is well known from the theory of primitive roots modulo p. Hence $D_x(a, 1)$ is divisible by p for $x = p - 1$ but for no smaller value of x, since obviously $a - 1$ is prime to p. But

$$D_{p-1}(a, 1) = \Pi F_d(a, 1),$$

where d ranges over all the divisors of $p - 1$ except unity. Now, $F_d(a, 1)$, $d < p - 1$, is not divisible by p; for, if so, $D_d(a, 1)$ would be divisible by p. But the first member of the above equation contains p as a factor and hence the second does also; and therefore $F_{p-1}(a, 1)$ is a multiple of p. That is, when p is prime an integer a exists such that $F_{p-1}(a, 1)$ is divisible by p.

Now let us suppose that $F_{p-1}(\alpha, \beta)$ is divisible by p. From Theorem XIV it follows that $F_\nu(\alpha, \beta)$ is divisible by p only when ν is of the form $\nu = (p - 1)p^k$. Hence $D_\nu(\alpha, \beta)$ is divisible by p only when ν is a multiple of $p - 1$, as is readily seen from equation (5). But (Theorem XIII) if $\lambda = \lambda_{\alpha\beta}(p)$, $D_\lambda(\alpha, \beta)$ is divisible by p. Hence $\lambda_{\alpha\beta}(p)$ is a multiple of $p - 1$. Since p is odd it follows at once from this and the definition of $\lambda_{\alpha\beta}(p)$ that p is prime. Therefore, if $F_{p-1}(\alpha, \beta)$ is divisible by the odd number p then p is prime. This completes the demonstration of the theorems above.

Owing to the difficulty of reckoning out the value of F_{p-1} in general the above theorems are not convenient in practice for the verification that a given number p is prime unless p is of special form. Like all other known tests for determining the character of a number as to being prime they are usually unwieldy for purposes of reckoning. But that in particular cases of interest they give tests which are remarkably easy of application is evidenced by the special results which we are now to deduce.

Let p be of the form

$$p = 2^{2^n} + 1, \qquad n > 1.$$

Then

$$F_{p-1}(\alpha, \beta) = \alpha^{\frac{p-1}{2}} + \beta^{\frac{p-1}{2}},$$

as one may readily show.

Let r be any odd prime of which p is a quadratic non-residue. Then, if p is prime, r is likewise a quadratic non-residue of p, as the theorem of quadratic reciprocity shows. Hence when p is prime we have

$$r^{\frac{p-1}{2}} + 1 \equiv 0 \bmod p.$$

Furthermore, from Theorem XXXIII and the value of F_{p-1} in the present case, it follows that p is prime when the above congruence is satisfied. Hence we have the following theorem:*

THEOREM XXXIV. *If* $p = 2^{2^n} + 1$, $n > 1$, *and* r *is any odd prime of which* p *is a quadratic non-residue, then a necessary and sufficient condition that* p *is prime is that*

$$r^{\frac{p-1}{2}} + 1 \equiv 0 \bmod p.$$

COROLLARY. *A necessary and sufficient condition that* $p = 2^{2^n} + 1$, $n > 1$, *is prime is that*

$$3^{\frac{p-1}{2}} + 1 \equiv 0 \bmod p.$$

Thus it may be verified readily that

$$3^{2^{15}} + 1 \equiv 0 \bmod 2^{16} + 1,$$

whence it follows that $2^{16} + 1$, $= 65{,}537$, is a prime number.

In testing a given number p of the form $2^{2^n} + 1$ as to its prime character one would reckon out successively the residues modulo p of the numbers

$$3^2,\ 3^{2^2},\ 3^{2^3},\ 3^{2^4},\ \cdots.$$

If the νth residue is -1 and if this is the first one which is -1, then p is prime if and only if $\nu = n$. Furthermore, from Theorem XXVI, corollary, it follows that when ν is not equal to n the divisors of the composite number p are of the form $2^\nu k + 1$.

As another special case let us assume p to be of the form

$$p = 2^{k+1} \cdot q + 1,$$

* This theorem and corollary should be compared with a related theorem due to Pepin (see Comptes rendus de l'académie des sciences, Paris, 85 (1877): 329–331).

where q is an odd prime. Then from equation (10) we have

$$F_{p-1} = \frac{\alpha^{2^k q} + \beta^{2^k q}}{\alpha^{2^k} + \beta^{2^k}}.$$

Hence from XXXIII we have the following theorem:

THEOREM XXXV. *If $p = 2^{k+1} \cdot q + 1$, where q is an odd prime, then a necessary and sufficient condition that p is prime is that an integer a exists such that*

$$\frac{a^{2^k q} + 1}{a^{2^k} + 1} \equiv 0 \bmod p.$$

COROLLARY. *A necessary and sufficient condition that $2^{k+1}3 + 1$ is prime is that an integer a exists such that*

$$a^{2^{k+1}} - a^{2^k} + 1 \equiv 0 \bmod 2^{k+1}3 + 1.$$

It is obvious that such special theorems as these may be obtained in unlimited number from our general results in XXXI to XXXIII. It is unnecessary to develop them further. It is, however, desirable to say a word in regard to the matter of actual verification of large primes by means of these theorems. For this purpose it will be convenient to speak briefly concerning the corollary above. Having selected a number a for trial one would reckon out successively the residues of

$$a,\ a^2,\ a^{2^2},\ \cdots,$$

say

$$\rho_0,\ \rho_1,\ \rho_2,\ \cdots,$$

where

$$\rho_i^2 \equiv \rho_{i+1} \bmod 2^{k+1} \cdot 3 + 1.$$

Each number ρ_i is thus readily obtained from the preceding one. Now if

$$\rho_{k+1} - \rho_k + 1 \equiv 0 \bmod 2^{k+1} \cdot 3 + 1,$$

then $2^{k+1} \cdot 3 + 1$ is prime. It is thus seen that when the reckoning is carried out in an appropriate manner it can be done with rapidity.

Thus in order to verify that $2^{41} \cdot 3 + 1$ is prime it would be sufficient to reckon out successively 41 residues modulo p, each one being determined from the preceding one by squaring and reducing modulo p,—if we suppose that an appropriate choice of a has been made. But it is only in special cases that we may be certain, in advance of the reckoning, that an appropriate choice of a has been made. Compare the corollary to Theorem XXXIV.

As a correlative of Theorems XXXI to XXXIII we have the following:

THEOREM XXXVI. *A sufficient condition that an odd number p is prime is that there exist relatively prime integers $\alpha + \beta$ and $\alpha\beta$ such that*

$$F_{p+1}(\alpha, \beta) \equiv 0 \bmod p.$$

The proof is analogous to that employed in the demonstration of a part of Theorem XXXI. When $F_{p+1}(\alpha, \beta)$ is divisible by p it follows from Theorem XIV that $F_\nu(\alpha, \beta)$ is divisible by p when and only when ν is of the form

$$\nu = (p+1)p^k.$$

Hence $D_\nu(\alpha, \beta)$ is divisible by p only when ν is a multiple of $p + 1$, as is readily seen from equation (5). But $D_\lambda(\alpha, \beta)$,

$$\lambda = \lambda_{\alpha\beta}(p),$$

is divisible by p, according to Theorem XIII; and therefore $\lambda_{\alpha\beta}(p)$ is a multiple of $p + 1$. It is obvious that this statement is true of an odd number p only when p is prime. Hence the theorem.

We shall now apply this general result to the determination of a necessary and sufficient condition that numbers of the form $2^n - 1$ shall be prime, thus obtaining as a corollary of our general theorem an important result which is due to Pepin (Comptes rendus de l'académie des sciences, Paris, 86 (1877): 307–310).

Suppose that $p = 2^n - 1$ is a prime number. Let r be a prime number of the form $4k + 1$ and write $r = a^2 + b^2$, where a and b are integers. Take

$$\alpha + \beta = 2a, \qquad \alpha\beta = a^2 + b^2,$$

whence

$$\alpha = a + b\sqrt{-1}, \qquad \beta = a - b\sqrt{-1}.$$

Suppose further that r is chosen so that p is a quadratic non-residue of r; then r is likewise a quadratic non-residue of p, as the law of quadratic reciprocity shows. Now Pepin (l. c., p. 307) has shown that if r is a quadratic non-residue of p, then

$$\alpha^{\frac{p+1}{2}} + \beta^{\frac{p+1}{2}} \equiv 0 \bmod p.$$

Applying this to the present case, we see that if p is prime it satisfies the relation

$$F_{p+1}(\alpha, \beta) \equiv 0 \bmod p.$$

But, according to the above theorem, if p satisfies this congruence it is prime. Hence we have the following corollary:

COROLLARY. *Let r be a prime number of the form $4k + 1$ of which $2^n - 1$ is a quadratic non-residue, and write $r = a^2 + b^2$ where a and b are integers. Put*

$$\alpha = a + b\sqrt{-1}, \qquad \beta = a - b\sqrt{-1}.$$

Then a necessary and sufficient condition that $2^n - 1$ is prime is that

$$F_{2^n} = \alpha^{2^{n-1}} + \beta^{2^{n-1}} \equiv 0 \bmod 2^n - 1.$$

It should be noticed, for purposes of reckoning, that we have the recurrence relation

$$F_{2^{k+1}} = F_{2^k}^2 - 2(\alpha\beta)^{2^k}$$

and that each of the terms of this equation represents an integer.

INDIANA UNIVERSITY,
November, 1912.

GEOMETRIC CHARACTERIZATION OF ISOGONAL TRAJECTORIES ON A SURFACE.*

By Joseph Lipka.

§ 1. The family of ∞^2 isogonal trajectories of a simple system of ∞^1 curves in the *plane* has formed the subject of several investigations. Cesàro† first stated the remarkable theorem that for the ∞^1 curves, of a family of ∞^2 isogonals, which pass through a given point, P, of the plane, the centers of curvature lie on a straight line (called the Cesàro line), or, in other words, the circles of curvature have another point, P', in common. Sheffers, in three papers,‡ discussed the same family of curves; in particular, he studied certain reciprocal families of isogonals arising from the point-to-point transformation of the plane set up by the correspondence of P and P' of the Cesàro theorem; also certain point-line transformations set up by the correspondence of a point P and its associated Cesàro line.

Kasner§ studied, for euclidean space, another class of curves, viz. natural families of curves, the extremals connected with variation problems of the type $\int F ds = minimum$, where F is any point function and ds is the element of arc in the space considered.‖ In the plane, he found that for such a family of ∞^2 curves, the same theorem as stated above for isogonals holds true. He then characterized a natural family by the additional property that for the ∞^1 curves of the family which pass through a given point of the plane, the two circles of curvature which there hyperosculate their respective curves are orthogonal. Now considering a family of ∞^2 curves as being composed of ∞^3 lineal elements, he showed that if in any isogonal family we rotate each lineal element about its initial point through a right angle, keeping its curvature unchanged, the new ∞^3 lineal elements form a natural family, thus characterizing a family of isogonals and completing the work of Cesàro and Sheffers.¶

* Presented to the American Mathematical Society, January 1, 1913.

† Lezioni di Geometria intrinsica, 1896; or Vorlesungen über Natürliche Geometrie, 1901, p. 148.

‡ Leipziger Berichte, 1898, pp. 261–294; ibid., 1904, pp. 105–116. Math. Annalen, 1905, vol. 60, pp. 491–531.

§ Trans. Am. Math. Soc., 2d Series, vol. 11 (1909), pp. 201–219.

‖ Among such extremals, the family of ∞^2 trajectories arising in a conservative field of force for the same constant of energy, s probably the most familiar example.

¶ Isogonals in the plane have also been characterized in a different way by W. M. Smith, in his dissertation, "Simply Infinite Systems of Plane Curves," 1912.

In a recent paper,* the author gave the geometric characterization of natural families of curves in a general curved space of n-dimensions, and briefly indicated that, in the case where $n = 2$, i. e., on an ordinary surface, the family of isogonals possesses one of the properties also possessed by a natural family. It is the purpose of this paper to give the *complete geometric characterization of isogonal trajectories on a surface*, and to exhibit the relations existing between isogonal and natural families of curves on any surface.†

§ 2. If we take an isothermal system of curves as parameter curves on the surface, we can write the element of arc length in the form

$$ds^2 = \lambda(u, v)[du^2 + dv^2]. \tag{1}$$

With this system of parameter curves, the angle, Ω, which a curve passing out in the direction dv/du makes with the v-parameter curve, becomes simply‡

$$\tan \Omega = \frac{dv}{du} = v'. \tag{2}$$

We can therefore represent any simple system of ∞^1 curves on the surface by

$$v' = \tan \Omega(u, v). \tag{3}$$

If we turn each lineal element u, v, v' about the point u, v through a constant angle α, we get

$$v' = \tan (\Omega + \alpha) \tag{4}$$

as the equation of the ∞^1 isogonals which cut (3) at a constant angle α To get the complete family of ∞^2 isogonals, we need merely differentiate once more and eliminate α, thus getting

$$v'' = (\Omega_u + \Omega_v v')(1 + v'^2) \qquad (\textit{type I}) \tag{5}$$

as the differential equation of the complete set of isogonals of the simple system (3).

On the other hand, starting out with the problem of finding the extremals connected with a variation problem of the type

$$\int F(u, v)ds = \int F\sqrt{\lambda(1 + v'^2)}du = \int H du = \text{minimum} \tag{6}$$

* Trans. Am. Math. Soc., 2d Series, vol. 13 (1912), pp. 77–95.

† As the previous paper contains only a statement (without proofs) of the theorems concerning natural families on an ordinary surface ($n = 2$), it has seemed best to embody here a fuller treatment of these curves also.

‡ Throughout the paper primes refer to total derivatives with respect to u; and literal subscripts to partial derivatives.

and applying the Euler condition for the vanishing of the first variation, viz.,

(7) $$H_v - H_{v'u} - v'H_{v'v} - v''H_{v'v'} = 0$$

we get

(8) $$v'' = [(\log F\sqrt{\lambda})_v - (\log F\sqrt{\lambda})_u v'](1 + v'^2) \qquad (\textit{type } N)$$

as the differential equation of the natural family of curves.

We notice that (5) and (8) are merely special forms of a more general type

(9) $$v'' = (\psi - \phi v')(1 + v'^2) \qquad (\textit{type } V)$$

which reduces to *type I* or to *type N* according to the restriction

(10′) $$\psi_v + \phi_u = 0$$

or

(10″) $$\psi_u - \phi_v = 0.$$

Equation (9) represents ∞^2 curves, of which ∞^1 pass through any given point, one in each direction on the surface.

§ 3. Consider a point P on the surface. The geodesic curvature and center of geodesic curvature of a curve through P are respectively the curvature and center of curvature of the orthogonal projection of the curve on the tangent plane to the surface at P. We shall take the point P as origin of coördinates, the tangent plane as the xy plane and the coördinate axes as the tangent lines to the u and v parapeter curves. Applying the formulas for the radius and center of geodesic curvature

(11) $$\frac{1}{\rho_g} = \frac{v'' - [(\log\sqrt{\lambda})_v - (\log\sqrt{\lambda})_u v'](1 + v'^2)}{\sqrt{\lambda}(1 + v'^2)^{\frac{3}{2}}}$$

and

(12) $$\xi = -\frac{v'}{\sqrt{1 + v'^2}}\rho_g, \qquad \eta = \frac{1}{\sqrt{1 + v'^2}}\rho_g$$

to the curves of *type V*, we get

(13) $$\frac{1}{\rho_g} = \frac{[\psi - (\log\sqrt{\lambda})_v] - [\phi - (\log\sqrt{\lambda})_u]v'}{\sqrt{\lambda}(1 + v'^2)}.$$

Now if we keep u, v fixed and allow v' to vary, the center of geodesic curvature will describe a certain locus; we get this by eliminating v' and ρ_g from equations (12) and (13), thus arriving at the *linear* equation

(14) $$[\phi - (\log\sqrt{\lambda})_u]\xi + [\psi - (\log\sqrt{\lambda})_v]\eta = \sqrt{\lambda}.$$

Hence we have the theorem:

Property A. *The locus of the centers of geodesic curvature of the* ∞^1 *curves of the type V (and therefore of type I or N) which pass through a given point, is a straight line.*

Conversely, if we have any system of ∞^2 curves on a surface, say

$$v'' = \chi(u, v, v'), \tag{15}$$

for which the centers of geodesic curvature of the curves through any point lie on a straight line, say

$$\alpha(u, v)\xi + \beta(u, v)\eta = \sqrt{\lambda(u, v)}, \tag{16}$$

we find, by introducing the values of ξ, η, ρ_g from (11) and (12), that

$$v'' = \{[\beta + (\log \sqrt{\lambda})_v] - [\alpha + (\log \sqrt{\lambda})_u]v'\}(1 + v'^2) \tag{17}$$

is the differential equation of the ∞^2 curves. This is of the *type V*. *Property A therefore characterizes the curves of type V.*

§ 4. For each curve c of *type V* which passes through a point P, we may draw the curve g which osculates (has 3-point contact with) c, and which has constant geodesic curvature (that of c at P) throughout. We call g the osculating geodesic circle of c. The question arises: in how many directions through P do the curves g hyperosculate (have 4-point contact with) the corresponding c curve? To answer this we need simply apply the condition

$$\frac{d(\rho_g)}{ds} = 0 \tag{18}$$

to the form (13), and we get

$$\alpha v'^2 + \beta v' + \gamma = 0, \tag{19}$$

where

$$\alpha = 4\lambda^2(\phi\psi - \phi_v) + 2\lambda\lambda_{uv} - 3\lambda_u\lambda_v,$$
$$\gamma = 4\lambda^2(\phi\psi - \psi_u) + 2\lambda\lambda_{uv} - 3\lambda_u\lambda_v,$$
$$\beta = 4\lambda^2[(\psi_v - \phi_u) - (\psi^2 - \phi^2)] + 2\lambda(\lambda_{uu} - \lambda_{vv}) - 3(\lambda_u^2 - \lambda_v^2).$$

Hence, as a consequence of property A, *the curves of type V are such that there are two directions through every point in which the osculating geodesic circles hyperosculate their corresponding curves.*

For the two directions given by (19) to be perpendicular, it is evidently both necessary and sufficient that we have $\alpha = \gamma$, i. e., $\phi_v - \psi_u = 0$, which is the condition (10″) that the curves of *type V* should form a natural family. Hence we have the theorem:

Property B. *For the curves of a natural family, the two directions through each point in which the osculating geodesic circles hyperosculate their corresponding curves, are orthogonal.*

But there is no such simple property which decides whether a system of *type V* is a system of isogonals.

§ 5. As was observed in § 1, we may, in the plane, start with any isogonal family, rotate each of its ∞^3 curvature elements about its initial point through a right angle, preserving its curvature, and thus get a new set of ∞^3 elements which form a natural family. The question arises whether a similar process will hold for any surface, replacing " curvature " by " geodesic curvature," of course.

For any curve of *type V* given by ϕ, ψ, in a direction v', we have from (13),

$$(20) \qquad \frac{1}{\rho_g} = \frac{[\psi - (\log \sqrt{\lambda})_v] - [\phi - (\log \sqrt{\lambda})_u]v'}{\sqrt{\lambda(1 + v'^2)}}.$$

For a curve of *type V* corresponding to $\bar{\phi}$, $\bar{\psi}$ and in a direction perpendicular to the first direction, i. e., in the direction $-(1/v')$, we have

$$(21) \qquad \frac{1}{\rho_g} = \frac{[\bar{\phi} - (\log \sqrt{\lambda})_u] + [\bar{\psi} - (\log \sqrt{\lambda})_v]v'}{\sqrt{\lambda(1 + v'^2)}};$$

for the equality of these geodesic curvatures, we must have

$$(22) \quad \bar{\phi} - (\log \sqrt{\lambda})_u = \psi - (\log \sqrt{\lambda})_v, \qquad \bar{\psi} - (\log \sqrt{\lambda})_v = -\phi + (\log \sqrt{\lambda})_u.$$

Now if the first system is isogonal, then $\psi_v = -\phi_u$, and if the second system is natural, then $\bar{\phi}_v = \bar{\psi}_u$; applying these relations to the quantities involved in (22) we get

$$(23) \qquad (\log \lambda)_{uu} + (\log \lambda)_{vv} = 0.$$

But f K denotes the Gaussian curvature of the surface, we have*

$$(24) \qquad K = -\frac{1}{2\lambda}[(\log \lambda)_{uu} + (\log \lambda)_{vv}].$$

Hence, *the only surfaces on which the above mentioned transformation is possible are the surfaces of zero Gaussian curvature, i. e., developable surfaces.*

We must therefore modify the above mentioned transformation. We may easily do so by not requiring the preservation of the geodesic curvature. Thus we may write (20) in the form

$$(25) \qquad \frac{1}{\rho_g} = \frac{\psi - \phi v'}{\sqrt{\lambda(1 + v'^2)}} - \frac{(\log \sqrt{\lambda})_v - (\log \sqrt{\lambda})_u v'}{\sqrt{\lambda(1 + v'^2)}}.$$

By comparison with (11) we see that the second term of the right member represents the geodesic curvature of that curve of the system $v'' = 0$ which passes out in the direction v'. Now the equation

* Bianchi, Vorlesungen über Differentialgeometrie, p. 68.

(26) $$v'' = 0\,,$$

is evidently the differential equation of the isogonals of a simple (∞^1) system of curves which form an isothermal set on the surface, i. e., corresponding to $\Omega = constant$. We must also note that (26) is also a natural family, corresponding to $F = \lambda^{-\frac{1}{2}}$. In general, *the only families on a surface which are both isogonal and natural are the isogonals of an isothermal system,* for we must have two restrictions on ϕ, ψ, viz.,

$$\phi_v - \psi_u = 0 \quad \text{and} \quad \phi_u + \psi_v = 0,$$

i. e., Ω satisfies the Laplacian

$$\Omega_{uu} + \Omega_{vv} = 0,$$

hence the curves (3) form an isothermal set. We might thus expect that the isogonals of an isothermal system, in their dual role, would lead to the solution of our problem.

In fact, if we subtract the geodesic curvatures of the family ϕ, ψ and the family $v'' = 0$, we get

(27) $$\frac{\psi - \phi v'}{\sqrt{\lambda(1 + v'^2)}}$$

and in order that this expression shall represent the curvature of the family $\bar{\phi}$, $\bar{\psi}$ in the perpendicular direction $-(1/v')$ as given by (21), we must have

(28) $$\bar{\phi} - (\log \sqrt{\lambda})_u = \psi, \qquad \bar{\psi} - (\log \sqrt{\lambda})_v = -\phi.$$

Then if the ϕ, ψ family is isogonal, i. e., $\psi_v = -\phi_u$, it follows immediately that $\bar{\phi}_v = \bar{\psi}_u$, i. e., the $\bar{\phi}$, $\bar{\psi}$ family is natural.

Defining *corresponding geodesic curvature elements* on a surface as two elements which have the same initial point and the same direction, we may state

PROPERTY C. *If from the geodesic curvature of the ∞^3 geodesic curvature elements composing an isogonal family, we subtract the geodesic curvature of the corresponding ∞^3 elements of the isogonals of an arbitrary isothermal system, and then rotate each element through a right angle, the ∞^3 new elements will form a natural family.**

We thus have a test for an isogonal family. *Given any ∞^2 curves on a surface, construct the isogonals of an arbitrary isothermal system. In a given direction through a given point there passes one geodesic curvature element of each family; construct a new element through this point and perpendicular to this direction, and whose geodesic curvature is equal to the difference of the*

* Of course the direction of rotation is the same for each element.

geodesic curvatures of the corresponding elements of the two families. If the new set of ∞^3 elements thus obtained possesses the characteristic properties A and B of a natural family, then the original family is a family of isogonals.

It is of interest to note that if, by this construction, the new set forms the family of geodesics, i. e., corresponding to $F = constant$, then the original family must have been the isogonals of an isothermal system; in other words, the isogonals of all isothermal systems on the surface are transformed into the family of geodesics.

The *analytic* curvature transformation which changes an isogonal family into a natural family is given by

$$(\Gamma)\qquad \begin{gathered} u = u_1, \quad v = v_1, \quad v' = -\frac{1}{v_1'}, \\ v'' = -\frac{v_1'' - [(\log\sqrt{\lambda})_v - (\log\sqrt{\lambda})_u v_1'][1 + v_1'^2]}{v_1'^3}. \end{gathered}$$

This transforms

$$(5)\qquad v'' = (\Omega_u + \Omega_v v')(1 + v'^2)$$

into

$$v'' = \{[\Omega_v + (\log\sqrt{\lambda})_v] - [\Omega_u + (\log\sqrt{\lambda})_u]v'\}(1 + v'^2)$$

which is of *type N* where

$$(29)\qquad (\log F)_v = \Omega_v, \quad (\log F)_u = \Omega_u, \quad \therefore F = e^{\Omega}.$$

We may easily verify that the transformation Γ is the analytical statement of the geometric transformation above described, and we thus have the following *corollary to Property C.*

The isogonal family corresponding to the point function $\Omega(u, v)$ is, by the theorem stated in property C, transformed into the natural family corresponding to the exponential of the same point function.

It may be easily verified that, except for developable surfaces, the operations Γ^2 and Γ^3 do not give isogonal or natural families, but that the operation Γ^4 does give the original isogonal family.

The reason for the simplicity of the transformation from isogonal to natural families for developable surfaces, lies in the fact that we may choose the straight line generators as our intermediary isothermal system, and these have for their isogonal trajectories the geodesics of the surface.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

ON THE SECOND VARIATION, JACOBI'S EQUATION AND JACOBI'S THEOREM FOR THE INTEGRAL.

$$\int F(x, y, x', y')dt.$$

BY A. DRESDEN.

Introduction.

In Weierstrass's theory of the integral

$$\int_{t_1}^{t_2} F(x, y, x', y')dt, \tag{1}$$

it is shown* that the second variation of (1) may be reduced to the form

$$\epsilon^2 \int_{t_1}^{t_2} (F_1 w'^2 + F_2 w^2)dt,$$

which is analogous to the form of the second variation of the integral

$$\int f\left(x, y, \frac{dy}{dx}\right) dx.$$

From this reduction of the second variation it follows that Jacobi's equation takes the form

$$F_2 u - (F_1 u')' = 0. \tag{2}$$

Then it is shown further, that when

$$x = x(t, \alpha), \qquad y = y(t, \alpha)$$

represent any one-parameter family of extremals for the integral (1), then

$$u = y'x_\alpha - x'y_\alpha$$

is a particular solution of equation (2).† This is Jacobi's theorem.

The purpose of this note is to obtain these results by means of a reduction of differential forms.‡ It appears to the author that in this way the derivation of the results becomes unified and less artificial.

* See, e. g., Bolza, Vorlesungen über Variationsrechnung, § 28, 29. Notations and terms used without explanation in this note are taken in the sense given them in that book.

† This statement of Jacobi's theorem is slightly more general than what is usually given; it is adopted so as to include such extensions as are made, for instance, in my paper on "The second derivatives of the extremal integral," Transactions of the American Mathematical Society, vol. 9, p. 471.

‡ See *Annals of Mathematics*, vol. 13 (second series), p. 149.

We assume that the function F in (1) satisfies all the conditions of the classical theory and that we have found an extremal represented by

$$(3) \qquad x = x(t), \qquad y = y(t), \qquad t_1 \leqq t \leqq t_2,$$

in which x and y are functions of class C''', which satisfy the initial conditions,

$$x(t_1) = x_1, \qquad y(t_1) = y_1, \qquad x(t_2) = x_2, \qquad y(t_2) = y_2,$$

and such that

$$(3a) \qquad [x'(t)]^2 + [y'(t)]^2 \neq 0, \qquad t_1 \leqq t \leqq t_2.$$

If now ξ and η are two arbitrary functions of t of class C'' on (t_1t_2) and vanishing at t_1 and at t_2, the second variation of the integral (1) may be put in the form

$$\delta^2 I = \epsilon^2 \int_{t_1}^{t_2} 2\Omega dt,$$

where

$$2\Omega \equiv F_{xx}\xi^2 + 2F_{xy}\xi\eta + F_{yy}\eta^2 + 2F_{xx'}\xi\xi' + 2F_{xy'}\xi\eta' + 2F_{x'y}\xi'\eta$$
$$+ 2F_{yy'}\eta\eta' + F_{x'x'}\xi'^2 + 2F_{x'y'}\xi'\eta' + F_{y'y'}\eta'^2,$$

the arguments of the partial derivatives of F being

$$x = x(t), \qquad y = y(t), \qquad x' = x'(t), \qquad y' = y'(t).$$

By means of Euler's theorem on homogeneous functions this reduces to

$$(4) \qquad \delta^2 I \equiv \epsilon^2 \int_{t_1}^{t_2} \left(\xi \frac{\partial\Omega}{\partial\xi} + \eta \frac{\partial\Omega}{\partial\eta} + \xi' \frac{\partial\Omega}{\partial\xi'} + \eta' \frac{\partial\Omega}{\partial\eta'} \right) dt,$$

in which expression the third and fourth terms may be integrated by parts, leading to

$$\partial^2 I \equiv \epsilon^2 \int_{t_1}^{t_2} (\xi\psi_1 + \eta\psi_2) dt,$$

where

$$\psi_1 \equiv \frac{\partial\Omega}{\partial\xi} - \frac{d}{dt}\frac{\partial\Omega}{\partial\xi'}, \qquad \psi_2 \equiv \frac{\partial\Omega}{\partial\eta} - \frac{d}{dt}\frac{\partial\Omega}{\partial\eta'}.$$

It is clear that ψ_1 and ψ_2 are linear differential forms of the second order in ξ and η, which satisfy the relation

$$(5) \qquad x'\psi_1 + y'\psi_2 = 0.$$

Further, if

$$x = x(t, \alpha), \qquad y = y(t, \alpha)$$

represent a one-parameter family of extremals, which for $\alpha = \alpha_0$ contains the particular extremal (3), which is under consideration, it follows upon direct substitution,* that

* As in the x-problem, see Bolza, l. c., p. 74.

$$\xi = x_a(t, \alpha_0), \qquad \eta = y_a(t, \alpha_0)$$

is a solution of the differential equations

(6) $$\psi_1 = 0, \qquad \psi_2 = 0.$$

These equations may therefore be looked upon as the analogues of Jacobi's equation in the x-problem.

To the differential forms

$$\begin{aligned}\psi_1 \equiv & - F_{x'x'}\xi'' - F_{x'y'}\eta'' - F'_{x'x'}\xi' + (F_{xy'} - F_{x'y} - F'_{x'y'})\eta' \\ & + (F_{xx} - F'_{xx'})\xi + (F'_{xy} - F'_{x'y})\eta, \\ \psi_2 \equiv & - F_{x'y'}\xi'' - F_{y'y'}\eta'' + (F_{x'y} - F_{xy'} - F'_{x'y'})\xi' - F'_{y'y'}\eta' \\ & + (F_{xy} - F'_{xy'})\xi + (F_{yy} - F'_{yy'})\eta,\end{aligned}$$

we now apply the tests for reducibility,* which in this case take the form:

(7)
$$\begin{vmatrix} F_{x'x'} & F_{x'y'} \\ F'_{x'x'} & F'_{x'y'} + F_{xy'} - F_{x'y} \end{vmatrix} = 0, \qquad \begin{vmatrix} F_{x'x'} & F_{x'y'} \\ F'_{xx'} - F_{xx} & F'_{xy'} - F_{xy} \end{vmatrix} = 0,$$
$$\begin{vmatrix} F_{x'y'} & F_{y'y'} \\ F'_{x'y'} - F_{xy'} + F_{x'y} & F'_{y'y'} \end{vmatrix} = 0, \qquad \begin{vmatrix} F_{x'y'} & F_{y'y'} \\ F'_{x'y} - F_{xy} & F'_{y'y} - F_{yy} \end{vmatrix} = 0.$$

To prove the first and third of these relations, we express every element of each of these determinants in terms of F_1 by means of formulæ, current in the theory.† To prove the second and fourth, we differentiate with respect to t the following formulae:‡

$$F_x \equiv x'F_{x'x} + y'F_{y'x}, \qquad F_y \equiv x'F_{x'y} + y'F_{y'y};$$

and, with respect to x' and y', the formula:§

$$F \equiv x'F_{x'} + y'F_{y'}.$$

This gives us:

(8) $$\begin{cases} x'(F'_{x'x} - F_{xx}) + y'(F'_{y'x} - F_{xy}) = 0, \\ x'F_{x'x'} \qquad\qquad + y'F_{y'x'} \qquad\qquad = 0, \end{cases}$$

and

(9) $$\begin{cases} x'(F'_{x'y} - F_{xy}) + y'(F_{y'y} - F_{yy}) = 0, \\ x'F_{x'y'} \qquad\qquad + y'F_{y'y'} \qquad\qquad = 0, \end{cases}$$

from which the desired relations follow, in view of condition 3(a).

* See *Annals of Mathematics*, vol. 13, p. 152.

† See Bolza, l. c., p. 196 (12a), p. 203 (I).

‡ Bolza, l. c., p. 196 (11).

§ Ibidem, (10).

Hence we conclude that ψ_1 and ψ_2 are reducible to the form

$$\psi_i \equiv A_i w'' + B_i w' + C_i w, \qquad (i = 1, 2)$$

where w is any linear function of ξ and η, $w = a\xi + b\eta$, such that

$$\begin{vmatrix} a & b \\ F_{x'y'} & F_{y'y'} \end{vmatrix} = 0.^*$$

Hence, we may put

$$w = -y'\xi + x'\eta,$$

and we can then determine A_i, B_i and C_i by undetermined coefficients. We find:

$$\begin{cases} y'A_1 = y'^2 F_1, \\ x'A_1 = x'y'F_1, \end{cases} \qquad \therefore\ A_1 = y'F_1,$$

$$\begin{cases} y'A_2 = -x'y'F_1, \\ x'A_2 = -x'^2 F_1, \end{cases} \qquad \therefore\ A_2 = -x'F_1,$$

$$\begin{cases} y'B_1 = y'^2 F_1', \\ x'B_1 = x'y'F_1', \end{cases} \qquad \therefore\ B_1 = y'F_1',$$

$$\begin{cases} y'B_2 = -x'y'F_1', \\ x'B_2 = -x'^2 F_1', \end{cases} \qquad \therefore\ B_2 = -x'F_1',$$

$$\begin{cases} x'C_1 = F_{xy} - F'_{x'y} - x'''y'F_1 - x''y'F_1', \\ y'C_1 = -F_{xx} + F'_{xx'} - y'''y'F_1 - y''y'F_1', \end{cases}$$

$$\begin{cases} y'C_2 = -F_{xy} + F'_{xy'} + y'''x'F_1 + y''x'F_1', \\ x'C_2 = F_{yy} - F'_{yy'} + x'''x'F_1 + x''x'F_1', \end{cases}$$

From Euler's differential equation for the integral (1), follows upon differentiation with respect to t:

$$F'_{x'y} - F'_{xy'} + y'(x''F_1)' - x'(y''F_1)' = 0,$$

and hence

$$F_{xy} - F'_{x'y} - x'''y'F_1 - x''y'F_1' \equiv F_{xy} - F'_{xy'} - y'''x'F_1 - y''x'F_1'.$$

By means of the same relation and formulæ (8_1) and (9_1) we verify:

* See *Annals*, l. c., p. 153. We notice in passing that $w(t_1) = w(t_2) = 0$ for all choices of a and b.

$$\begin{cases} x'(+F'_{xx'} - F_{xx} - y'''y'F_1 - y''y'F_1') \\ \qquad\qquad + y'(+F'_{x'y} - F_{xy} + x'''y'F_1 + x''y'F_1') = 0, \\ x'(F'_{xy'} - F_{xy} + y'''x'F_1 + y''x'F_1') \\ \qquad\qquad + y'(F'_{yy'} - F_{yy} - x'''x'F_1 - x''x'F_1') = 0. \end{cases}$$

Hence there exists a function F_2,* such that

$$F'_{xx'} - F_{xx} - y'''y'F_1 - y''y'F_1' = -y'^2F_2,$$

$$F'_{x'y} - F_{xy} + x'''y'F_1 + x''y'F_1' = F'_{xy'} - F_{xy} + y'''x'F_1 + y''x'F_1'$$
$$= +x'y'F_2,$$

$$F'_{yy'} - F_{yy} - x'''x'F_1 - x''x'F_1' = -x'^2F_2.$$

Consequently, we have:

$$C_1 = -y'F_2,$$

and

$$C_2 = +x'F_2.$$

We can now write

$$\begin{cases} \psi_1 \equiv y'(F_1w'' + F_1'w' - F_2w), \\ \psi_2 \equiv -x'(F_1w'' + F_1'w' - F_2w). \end{cases}$$

From this result, we conclude:

1. The second variation of the integral (1) may be reduced to the form:

$$\delta^2 I \equiv \epsilon^2 \int_{t_1}^{t_2} (y'\xi - x'\eta)(F_1w'' + F_1'w' - F_2w)dt.$$

Since w vanishes at t_1 and t_2, we may add to this the integral

$$\epsilon^2 \int_{t_1}^{t_2} \frac{d}{dt}(ww'F_1)dt \equiv \epsilon^2 \int_{t_1}^{t_2} (ww'F_1' + ww''F_1 + w'^2F_1)dt,$$

which gives us the expression:

$$\delta^2 I \equiv \epsilon^2 \int_{t_1}^{t_2} (F_2w^2 + F_1w'^2)dt,$$

known from the current theory.

2. Jacobi's differential equation for the integral (1) reduces to

$$(2) \qquad F_1w'' + F_1'w' - F_2w = 0,$$

since

$$x'^2 + y'^2 \neq 0,$$

which is known as Weierstrass's form of Jacobi's equation.

* It can be verified without much difficulty that the function F_2 here defined is identical with the function introduced by Weierstrass.

3. Since the functions

$$\xi = x_a(t, \alpha_0), \qquad \eta = y_a(t, \alpha_0)$$

satisfy equations (6), it follows that the function:

$$w = x' y_a - y' x_a$$

will satisfy the Weierstrass form of this equation,

$$x = x(t, \alpha), \qquad y = y(t, \alpha)$$

representing any one-parameter family of extremals: Weierstrass's form of Jacobi's theorem.

The extension of this method to the treatment of the second variation in the more general problems of the calculus of variations, which seems to be possible, has not as yet been obtained.

University of Wisconsin,
February, 1913.

QUARTIC SURFACES INVARIANT UNDER PERIODIC TRANSFORMATIONS.

By Professor F. R. Sharpe and Dr. F. M. Morgan.

In 1845 Steiner* stated the following theorem: "Let P and Q be two fixed points on a plane cubic curve (or double points on a plane quartic curve) and A a variable point on the curve. Let PA meet the curve again in A_1, QA_1 in A_2, PA_2 in A_3, $\cdots$, QA_{2n-1} in A_{2n}. If A_{2n} coincides with A for one position of A, then it coincides with A for every position of A." In 1910 Snyder† considered a quartic surface having two conical points P and Q, and stated the condition that the two transformations A into A_1 and A_1 into A_2 should be commutative for the section of the quartic surface by any plane through the line PQ. The double transformation A into A_2 is then of period two. That is if S is the first transformation and T the second, then $(ST)^2 = 1$. This suggested to the late Professor J. E. Wright, of Bryn Mawr, the problem of finding quartic surfaces such that $(ST)^3 = 1$. His untimely death prevented him from solving the problem. Professor Snyder, of Cornell, recently proposed it to us, and the solution is given in this paper. It may also be interpreted as the condition that the two involutorial transformations S and T of the general (2, 2) correspondence satisfy the condition $(ST)^3 = 1$.

The above theorem of Steiner follows easily from the expression of the coördinates of any point on a non-singular cubic curve in terms of elliptic functions $p(u)$ of a parameter u

$$x_1 = \rho p'(u), \qquad x_2 = \rho p(u), \qquad x_3 = \rho.$$

It is well known that the coördinates can be so chosen that the sum of the parameters of three collinear points on the curve is equal to a sum of the multiples of the periods $2\omega_1$, $2\omega_2$.

Denoting the parameter of a point by the corresponding small letter, we have (mod $2\omega_1$, $2\omega_2$)

$$p + a + a_1 \equiv 0,$$

$$q + a_1 + a_2 \equiv 0.$$

Therefore

$$p - q + a - a_2 \equiv 0.$$

* Crelle, vol. 32 (1845), pp. 182–184.

† Trans. Am. Math. Soc., vol. 11, p. 16, Sturm, Geo., Verwandtschaften, Band I, p. 267.

Similarly

$$p - q + a_2 - a_4 \equiv 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$p - q + a_{2n-2} - a_{2n} \equiv 0.$$

Hence if A_{2n} coincides with A, by addition we find

$$n(p - q) \equiv 0$$

which is independent of the position of A. The parameters of P and Q are seen to differ by one nth of a period.

If we invert with respect to a triangle PQR, where R is a point not on the cubic, the lines through P and Q are inverted into lines through P and Q, but the cubic is inverted into a quartic having P and Q for double points, so the theorem holds in the latter case.

The condition for periodicity may be expressed in a simple geometric form by taking the limiting case of the theorem as A approaches P. For a cubic curve and period two, A_3 is the point where the line PQ again meets the curve. Also PA and QA_2 are the tangents at the points P and Q respectively. The condition for per od two is therefore that these tangents meet on the curve at the point A_1. For period three A_5 is the point where PQ again meets the curve and PA and QA_4 are the tangents at P and Q respectively. If these tangents meet the curve again in A_1 and A_3, then the condition is that the lines QA_1 and PA_3 meet on the curve in the point A_2.*

For a quartic curve with double points P and Q and period two, PA and PA_3 are the tangents at P, also A_1 and A_2 are their points of intersection with the curve. The condition therefore is that the points QA_1A_2 are collinear.

For period three the tangents PA and PA_5 at P meet the curve in two points A_1, A_4 such that QA_1 and QA_4 meet the curve in two points A_2 and A_3 which are collinear with P.

We will now proceed to express these conditions analytically. Using homogeneous coördinates $x_1x_2x_3x_4$, the equation of a quartic surface having conical points at

$$P = (0, 0, 0, 1) \quad \text{and} \quad Q = (0, 0, 1, 0)$$

is of the form

(1) $(a_1x_3^2 + b_1x_3 + c_1)x_4^2 + (a_2x_3^2 + b_2x_3 + c_2)x_4 + (a_3x_3^2 + b_3x_3 + c_3) = 0$

or

(2) $(a_1x_4^2 + a_2x_4 + a_3)x_3^2 + (b_1x_4^2 + b_2x_4 + b_3)x_3 + (c_1x_4^2 + c_2x_4 + c_3) = 0,$

where the coefficients are homogeneous functions of x_1 and x_2 such that the equations are homogeneous and of degree four in the coördinates.

* Crelle, vol. 32, pp. 182–184.

The form (1) shows that the transformation S interchanges the points (x_1, x_2, x_3, x_4) and (x_1, x_2, x_3, x_4') where x_4 and x_4' are the roots of the quadratic (1) in x_4. The form (2) similarly shows that the transformation T interchanges the points (x_1, x_2, x_3, x_4) and (x_1, x_2, x_3', x_4) where x_3 and x_3' are the roots of the quadratic (2) in x_3.

If we keep x_1/x_2 fixed, we have the section of the quartic surface by a plane through P and Q. This section has double points at P and Q and has for tangents at P

$$a_1x_3^2 + b_1x_3 + c_1 = 0. \tag{3}$$

First the analytic condition for period two will be deduced. Denoting the roots of (3) by x_3, x_3' the tangents at P meet the surface at

$$A_1 = (x_1, x_2, x_3, x_4)$$

and

$$A_2 = (x_1, x_2, x_3', x_4')$$

where from (1)

$$x_4 = -\frac{a_3x_3^2 + b_3x_3 + c_3}{a_2x_3^2 + b_2x_2 + c_2}, \qquad x_4' = -\frac{a_3x_3'^2 + b_3x_3' + c_3}{a_2x_3'^2 + b_2x_2' + c_2}. \tag{4}$$

Hence, by subtracting and dividing by $x_3 - x_3'$, we have

$$\frac{x_4 - x_4'}{x_3 - x_3'} = \frac{C_1x_3x_3' - B_1(x_3 + x_3') + A_1}{(a_2x_3x_3' - c_2)^2 + [b_3x_3x_3' + c_2(x_3 + x_3')][b_2 + a_2(x_3 + x_3')]}, \tag{5}$$

where the large letters denote the cofactors of the corresponding small letters in the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{6}$$

But from (3)

$$a_1x_3x_3' = c_1 \qquad \text{and} \qquad a_1(x_3 + x_3') = -b_1.$$

Hence (5) becomes

$$\frac{x_4 - x_4'}{x_3 - x_3'} = \frac{a_1\Delta}{r}, \tag{7}$$

where

$$r = B_3^2 - A_3C_3.$$

If the transformation is of period two we proved that the points QA_1A_2 must be collinear. Hence $x_4 = x_4'$ and therefore

$$\Delta = 0.* \tag{8}$$

* Sturm, Geo., Verwandtschaften, Band I, p. 267.

This is therefore the condition that the transformation be of period two for the section considered. It is of degree six in x_1 and x_2. There are therefore in general six planes through PQ which satisfy the relation $(ST)^2 = 1$. If however the seven coefficients of this equation are all zero, then all sections through PQ will satisfy the relation $(ST)^2 = 1$. The twenty-seven coefficients of (1) must therefore satisfy seven conditions in order that the surface may be invariant under a transformation ST such that $(ST)^2 = 1$.

For period three the old A_2 becomes A_4 while the new

$$A_2 = (x_1, x_2, x_3'', x_4)$$

and

$$A_3 = (x_1, x_2, x_3''', x_4').$$

From (2) follow the relations

$$\text{(9)} \qquad \begin{aligned} x_3 + x_3'' &= -\frac{b_1x_4^2 + b_2x_4 + b_3}{a_1x_4^2 + a_2x_4 + a_3}, \\ x_3' + x_3''' &= -\frac{b_1x_4'^2 + b_2x_4' + b_3}{a_1x_4'^2 + a_2x_4' + a_3}. \end{aligned}$$

But A_2, A_3 and P are collinear. Therefore $x_3'' = x_3'''$.
Hence from (9) it follows that

$$\text{(10)} \qquad \frac{x_3 - x_3'}{x_4 - x_4'} = \frac{C_3x_4x_4' - C_2(x_4 + x_4') + C_1}{(a_1x_4x_4' - a_3)^2 + [a_2x_4x_4' + a_3(x_4 + x_4')][a_2 + a_1(x_4 + x_4')]}$$

Now from (4) it can be shown that

$$x_4 + x_4' = \frac{2B_2B_3 - A_2C_3 - A_3C_2}{B_3^2 - A_3C_3}$$

and

$$x_4x_4' = \frac{B_2^2 - A_2C_2}{B_3^2 - A_3C_3}.$$

For brevity let

$$2B_2B_3 - A_2C_3 - A_3C_2 = s; \qquad B_2^2 - A_2C_2 = q;$$

$$2B_1B_3 - A_1C_3 - A_3C_1 = t; \quad 2B_1B_2 - A_1C_2 - A_2C_1 = u; \quad B_1^2 - A_2C_1 = p.$$

Then

$$x_4 + x_4' = \frac{s}{r}$$

and

$$x_4x_4' = \frac{q}{r}.$$

By substituting in (10) we have

$$\frac{x_3 - x_3'}{x_4 - x_4'} = \frac{(C_3q - C_2s + C_1r)r}{(a_1q - a_3r)^2 + (a_2q + a_3s)(a_2r + a_1s)}. \tag{11}$$

If then between (6) and (11) the ratio $\dfrac{x_3 - x_3'}{x_4 - x_4'}$, be eliminated, we have

$$\frac{r}{a_1\Delta} = \frac{(C_3q - C_2s + C_1r)r}{(a_1q - a_3r)^2 + (a_2q + a_3s)(a_2r + a_1s)} \tag{12}$$

and hence

$$a_1\Delta(C_3q - C_2s + C_1r) = (a_1q - a_3r)^2 + (a_2q + a_3s)(a_2r + a_1s). \tag{13}$$

Using the identity

$$a_2^2q + a_2a_3s + a_3^2r \equiv a_1^2p + a_1C_1\Delta$$

and dividing out a_1 as a factor we have

$$\Delta(C_3q - C_2s) = a_1(q^2 + pr) - 2a_3qr + a_2qs + a_3s^2.$$

Now using the identities

$$a_3s + 2a_2q \equiv - a_1u - C_2\Delta,$$

$$a_2s + 2a_3r \equiv - a_1t - C_3\Delta,$$

and dividing out a_1 as a factor, we have

$$q^2 + pr - su + qt = 0. \tag{14}$$

This is the condition that the two involutorial transformations S and T of the general (2 2) correspondence (1) satisfy $(ST)^3 = 1$.

It is of degree sixteen in x_1 and x_2. Hence there are in general sixteen sections of (1) by planes through PQ such that the condition $(ST)^3 = 1$ is satisfied. If however the twenty-seven coefficients of (1) are such that the seventeen coefficients of (14) are all zero, then all sections through PQ will satisfy the relation $(ST)^3 = 1$.

A similar method applies to period four, but the degree of the condition found is greater than twenty-six in x_1 and x_2, hence it seems doubtful that there exist quartic surfaces invariant under this type of transformation.

The condition given in (14) remains true in all cases but the proof given appears to fail when $a_1 = 0$. If we keep x_1/x_2 fixed as before, we have the section of the quartic surface by a plane through P and Q. This quartic section degenerates into a cubic and the line PQ. The tangent PA at P to the cubic, is

$$x_3 = -\frac{c_1}{b_1}. \tag{15}$$

This meets the cubic again in

$$A_1 = \left(x_1, x_2, \frac{-c_1}{b_1}, x_4\right)$$

while the line QA_1 is

$$x_4 = -\frac{a_3x_3^2 + b_3x_3 + c_3}{a_2x_3^2 + b_2x_3 + c_2}. \tag{16}$$

The other tangent at P in the general case, namely PA_4, degenerates as $a_1 \doteq 0$, into the line PQ, but in such a way that the tangent at Q to the cubic meets it again in A_3, QA_3 being

$$x_4' = \frac{-a_3}{a_2}. \tag{17}$$

If QA_1 meets the cubic in

$$A_2 \equiv (x_1, x_2, x_3'', x_4),$$

then from (1) and (2)

$$\frac{-b_1}{c_1} + \frac{1}{x_3''} = -\frac{b_1x_4^2 + b_2x_4 + b_3}{c_1x_4^2 + c_2x_4 + c_3}. \tag{18}$$

and if

$$A_3 \equiv (x_1, x_2, x_3''', x_4'),$$

then

$$\frac{1}{x_3'''} = -\frac{b_1x_4'^2 + b_2x_4' + b_3}{c_1x_4'^2 + c_2x_4' + c_3}. \tag{19}$$

But PA_2A_3 are collinear, hence $x_3'' = x_3'''$. We therefore have

$$\frac{b_1}{c_1} = \frac{(x_4 - x_4')[A_3x_4x_4' - A_2(x_4 + x_4') + A_1]}{(c_1x_4x_4' - c_3)^2 + [c_2x_4x_4' + c_3(x_4 + x_4')][c_2 + c_1(x_4 + x_4')]}. \tag{20}$$

But from (15), (16), and (17)

$$x_4 - x_4' = \frac{b_1\Delta}{r}, \tag{21}$$

where x_4 and x_4' are the roots of

$$rx_4^2 - sx_4 + q = 0. \tag{22}$$

Hence (20) becomes

$$C_1\Delta(A_3q - A_2s + A_1r) = (c_1q - c_3r)^2 + (c_2r + c_3s)(c_2r + c_1s).$$

This differs from (13) only in having a's instead of c's and therefore leads to the same result (14) as this condition is unaltered when these letters are interchanged.

This method also fails when $c_1 = 0$, but corresponding to the equations

of the last case, we have

(15′) $$x_3 = 0,$$

(16′) $$x_4 = \frac{-c_3}{c_2},$$

(17′) $$x_4' = \frac{-a_3}{a_2},$$

(18′) $$x_3'' = -\frac{b_1x_4^2 + b_2x_4 + b_3}{a_2x_4 + a_3},$$

(19′) $$\frac{1}{x_3'''} = -\frac{b_1x_4'^2 + b_2x_4' + b_3}{c_2x_4' + c_3}.$$

Hence the condition $x_3'' = x_3'''$ leads to

$$(b_1q - b_3r)^2 + (b_2q + b_3s)(b_1s + b_2r) = 0,$$

which when it is transformed as in the previous cases and the factor b_1^2 is divided out, gives as before

(14) $$q^2 + pr - su + qt = 0.$$

If $a_1 = c_1 = a_3 = 0$, then $q = 0$ and the condition (14) reduces to

$$(a_2c_2 - b_1b_3)c_3^2 - b_2b_3c_2c_3 + b_3^2c_2^2 = 0.$$

The planes $x_3 = 0$ and $x_4 = 0$ are now tangent planes at P and Q respectively.

If $c_3 = c_2x_2$, then (14) becomes

$$(a_2c_2 - b_1b_3)x_2^2 - b_2b_3x_2 + b_3^2 = 0$$

and if $c_2 = b_3$, then

$$(a_2 - b_1)x_2^2 - b_2x_2 + b_3 = 0.$$

Equation (1) now takes the form

$$b_1(x_3x_4^2+x_3x_2^2)+a_2(x_3^2x_4-x_3x_2^2)+b_2(x_3x_4+x_2x_3)+c_2(x_2+x_3+x_4)=0.$$

This appears to be the simplest type of surface that fulfils the condition (14). It contains eleven arbitrary constants.

Let K_n' denote a cone of order n with vertex at (0, 0, 0, 1) and K_n'' also a cone of order n with vertex at (0, 0, 1, 0).

Equation (1) may then be written in the form

(20) $$K_2'x_4^2 + K_3'x_4 + K_4' = 0$$

and equation (2)

(21) $$K_2''x_3^2 + K_3''x_3 + K_4'' = 0.$$

Therefore the transformation S is

$$(22)\qquad \begin{aligned} x_1 &= x_1'K_2', & x_3 &= x_3'K_2', \\ x_2 &= x_2'K_2', & x_4 &= -\,x_4'K_2' - K_3' \end{aligned}$$

This is a transformation of monoidal type.* The image of any plane not passing through (0, 0, 0, 1) is a cubic surface with a conical point at (0, 0, 0, 1), the image of this point being $K_2' = 0$. The fundamental curves are the six lines $K_2' = 0$, $K_3' = 0$. Similarly T is

$$(23)\qquad \begin{aligned} x_1' &= x_1''K_2'', & x_2' &= x_2''K_2'', \\ x_3' &= -\,x_3''K_2'' - K_3'', & x_4' &= x_4''K_2''. \end{aligned}$$

Hence ST is

$$(24)\qquad \begin{aligned} x_1 &= x_1''K_2''[a_1(x_3''K_2'' + K_3'')^2 - b_1(x_3''K_2'' + K_3'')K_2'' + c_1K_2''^2] \\ &= x_1''K_2''F_6, \\ x_2 &= x_2''K_2''F_6, \\ x_3 &= -\,(x_3''K_2'' + K_3'')F_6, \\ x_4 &= -\,(x_4''F_6 + F_7)K_2'', \end{aligned}$$

where

$$F_7 = a_2(x_3''K_2'' + K_3'')^2 - b_2(x_3''K_2'' + K_3'')K_2'' + c_2K_2''^2.$$

This is a Cremona transformation. The image of any plane not passing through (0, 0, 0, 1) nor (0, 0, 1, 0) is a surface of degree nine, having a six fold point at (0, 0, 1, 0) and a conical point at (0, 0, 0, 1). The images of these points are $F_6 = 0$ and $K_2'' = 0$ respectively.

The sum of the degrees of the fundamental curves† is $9^2 - 9 = 72$. These curves are the six lines $K_2'' = 0$, $K_3'' = 0$ counted nine times, and the six cubics into which T transforms the six lines $K_2' = 0$, $K_3' = 0$.

Hence ST transforms the plane sections of (20) into curves of degree 36 passing four times through (0, 0, 0, 1) and twelve times through (0, 0, 1, 0); similarly for TS. Since ST is of period three it follows that the first triply infinite linear system of curves is transformed by ST into the second system, which is also of degree 36, but passes twelve times through (0, 0, 0, 1) and four times through (0, 0, 1, 0).

We may also write (22) in the form

$$(25)\qquad \begin{aligned} x_1 &= x_1'x_4'K_2', & x_2 &= x_2'x_4'K_2', \\ x_3 &= x_3'x_4'K_2', & x_4 &= K_4'. \end{aligned}$$

This is also a monoidal transformation but of degree four, the point

* Doehlemann, Geometrischen Transformationen, Band II, Art. 167.

† Sturm, Geo., Verwandtschaften, Band IV, p. 341.

(0, 0, 0, 1) being a triple point. Its image is $x_4'K_2' = 0$. The fundamental curves are of degree $4^2 - 4 = 12$ and consist of the eight lines $K_2' = 0$, $K_4' = 0$ and the plane quartic $x_4' = 0$, $K_4' = 0$. T is a similar transformation. Forming the product ST we find a Cremona transformation of degree 13 for which (0, 0, 0, 1) is a triple point and (0, 0, 1, 0) a nine fold point. The sum of the degrees of the fundamental curves is $13^2 - 13 = 156$. The curves are (*a*) the eight quartics into which T sends the eight lines $K_2' = 0$, $K_4' = 0$; (*b*) the fundamental curves of T counted nine times; (*c*) the three lines $x_3''K_2'' = 0$, $x_4'' = 0$; (*d*) the line $x_1 = 0$, $x_2 = 0$ counted three times.

A plane section of (20) is transformed into a variable curve of degree 36 as before, together with the fixed curves, (*a*) $x_4 = 0$ and (*b*) $x_3 = 0$ counted three times, thus making the total degree 52.

We may also consider ST as the product of a cubic and quartic transformation and thus find similar results. These transformations all have the same meaning for points on the quartic surface, but are distinct for other points.

Cornell University and Dartmouth College,
December, 1912.

POSTULATE-SETS FOR ABELIAN GROUPS AND FIELDS.

By Wallie Abraham Hurwitz.

Postulate-sets for fields have been given by Dickson* and Huntington;† these have naturally been founded on previous definitions of abelian groups. Several years ago the writer gave a postulate-set for abelian groups‡ which not only (like a set of Huntington's on which it was based) made no assumption of closure with respect to the group operation, but also replaced both the associative and commutative laws by a single law of somewhat the same character as the associative law. The present paper contains a slightly altered form of this definition and an extension to the case of fields. The postulate-set for fields contains five postulates, aside from one specifying the cardinal number of the class,—thus reducing by two postulates (one each for addition and multiplication) the smallest number used hitherto. As an incidental result a new type of finite linear algebras is found, obeying the commutative law and admitting unique division by non-zero elements, but not obeying the associative law or containing an idemfacient element.§ It need scarcely be mentioned that a definition of groups or fields which introduces closure, associativity, and commutativity as theorems rather than as postulates is justified by its logical interest, and not by any practical use.

The proofs in the paper are complete in themselves and may be read independently of previous postulate-sets for groups and fields.

1. Definition of Abelian Groups. A class K with an operation $\circ$ between pairs of elements will be called an abelian group if the following postulates hold:

(1) *If a, b, c, $a \circ b$, $c \circ b$, and $a \circ (c \circ b)$ belong to K, then* $(a \circ b) \circ c = a \circ (c \circ b)$.

(2) *If a and b belong to K, there is an element x of K such that* $a \circ x = b$.

The number of elements is specified by adding one of the postulates:

(N_n) *K contains n elements.*

(N') *K is countably infinite.*

(N'') *K has the cardinal number of the continuum.*

---

* Transactions of the American Mathematical Society, vol. 4 (1903), p. 13; vol. 6 (1905), p. 108; Göttinger Nachrichten (1905), p. 358.

† Transactions of the American Mathematical Society, vol. 4 (1903), p. 31; vol. 6 (1905), p. 181.

‡ These *Annals*, Second Series, vol. 8 (1907), p. 94.

§ I. e., an element whose product with any element reproduces the latter.

Every abelian group satisfies (1) and (2), and abelian groups exist satisfying each of (N_n) (for any positive integer n), (N'), (N''). It will be shown that conversely the ordinary properties of an abelian group follow from (1), (2).

In the first place, if a, b, and either $a \circ b$ or $b \circ a$ belong to K, then

$$a \circ b = b \circ a.$$

Suppose, for example, that $a \circ b$ belongs to K. By (2) take x so that $a \circ x = b$; then by (1),

$$a \circ (a \circ x) = (a \circ x) \circ a, \quad \text{or} \quad a \circ b = b \circ a.$$

We now show, with the aid of this result, that for any two elements a, b of K, $a \circ b$ is an element of K. By (2) take successively x, y, z, w, so that

$$b \circ x = a, \quad a \circ y = x, \quad y \circ z = a, \quad a \circ w = z.$$

Then by (1),

$$a = y \circ z = y \circ (a \circ w) = y \circ (w \circ a) = (y \circ a) \circ w = x \circ w,$$

so that

$$b \circ x = a = x \circ w = w \circ x,$$

and

$$z = a \circ w = w \circ a = w \circ (b \circ x) = (w \circ x) \circ b = a \circ b;$$

therefore $a \circ b$ is the element z known to belong to K.

Since $a \circ b$ belongs to K and the commutative law holds, we see from (1) that the associative law holds; these results, with (2), comprise one of the ordinary forms of definition of abelian groups.

The postulates (1), (2), with any of (N_n) $(n > 1)$,* (N'), (N'') are independent. The proof will appear as part of the corresponding work for fields.

2. Definition of Fields. A class K, with two operations $\oplus$ and $\otimes$ between pairs of elements, will be called a field if the following postulates hold:

(*A*1) *If* a, b, c, $a \oplus b$, $c \oplus b$, *and* $a \oplus (c \oplus b)$ *belong to* K, *then*

$$(a \oplus b) \oplus c = a \oplus (c \oplus b).$$

(*A*2) *If a and b belong to K, there is an element x of K such that*

$$a \oplus x = b.$$

(*M*1) *If* a, b, c, $a \otimes b$, $c \otimes b$, *and* $a \otimes (c \otimes b)$ *belong to* K, *then*

$$(a \otimes b) \otimes c = a \otimes (c \otimes b).$$

* If $n = 1$, postulate (2) is sufficient.

($M2$) *If a and b belong to K and $a \oplus a \neq a$, there is an element x of K such that*

$$a \otimes x = b.$$

(D) *If $a, b, c, a \otimes b, a \otimes c, b \oplus c$, and $(a \otimes b) \oplus (a \otimes c)$ belong to K, then*

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c).$$

We specify the number of elements by adding one of the postulates (N_n) ($n > 1$), (N'), (N'') of the preceding section.

Postponing inquiry as to whether (or when) the postulates are consistent and independent, we proceed to show that the concept defined by ($A1$), ($A2$), ($M1$), ($M2$), (D) agrees with the usual notion of a field. Any field in the ordinary sense satisfies ($A1$), ($A2$), ($M1$), ($M2$), (D); we have then to show that conversely these postulates imply the field properties.

Since ($A1$), ($A2$) are precisely (1), (2) with $\oplus$ for $\circ$, we have at once:

THEOREM I. *K is an abelian group with respect to $\oplus$.*

Among the consequences of this theorem we note particularly: when a and b belong to K, then $a \oplus b$ belongs to K; one and only one element of K (the "identity" of the group, which we may call 0) satisfies the condition

$$a \oplus a = a;$$

from

$$a \oplus x = a$$

follows $x = 0$ and conversely.

LEMMA. *If a and $a \otimes 0$ belong to K, then $a \otimes 0 = 0$.*

In case $a = 0$, we have by (D)

$$0 \otimes (0 \oplus 0) = (0 \otimes 0) \oplus (0 \otimes 0)$$

or

$$(0 \otimes 0) \oplus (0 \otimes 0) = 0 \otimes 0,$$

whence

$$0 \otimes 0 = 0.$$

Suppose then $a \neq 0$. By ($A2$) choose x so that

$$(a \otimes 0) \oplus x = 0,$$

and by ($M2$) choose y so that

$$a \otimes y = x.$$

Then by (D),

$$a \otimes (0 \oplus y) = (a \otimes 0) \oplus (a \otimes y)$$

or

$$x = (a \otimes 0) \oplus x,$$

and therefore

$$a \otimes 0 = 0.$$

If we now denote by K' the class obtained by removing from K the element 0, we have:

THEOREM II. *K' is an abelian group with respect to* $\otimes$.

Since any element of K' is an element of K, $(M1)$ implies (1) with $\otimes$ for $\circ$ and K' for K. Also the hypothesis of $(M2)$ implies the hypothesis of (2); in order to see that the conclusion of (2) agrees with the conclusion of $(M2)$ we need only ascertain that the x defined is in K', i. e., that $x \neq 0$. This is assured by the preceding lemma; for the hypothesis $a \otimes x = b$, $x = 0$ would imply $b = 0$, so that b would not be in K'. As a consequence we note: if a and b belong to K', then $a \otimes b$ belongs to K'; or, in other words, if a and b belong to K and $a \neq 0$, $b \neq 0$, then $a \otimes b$ belongs to K and $a \otimes b \neq 0$.

THEOREM III. *If a belongs to K, then* $a \otimes 0 = 0 \otimes a = 0$.

First suppose $a \neq 0$. By $(M2)$ take x so that

$$a \otimes x = 0;$$

theorem II shows that the assumption $x \neq 0$ would involve

$$a \otimes x \neq 0,$$

which is untrue; hence $x = 0$ and $a \otimes 0 = 0$. By $(M1)$,

$$(a \otimes 0) \otimes a = a \otimes (a \otimes 0);$$

hence

$$0 \otimes a = a \otimes 0 = 0.$$

Consider now the case $a = 0$. Choose $x \neq 0$. By $(A2)$ take y so that

$$x \oplus y = 0;$$

evidently $y \neq 0$. Then by (D),

$$0 \otimes (x \oplus y) = (0 \otimes x) \oplus (0 \otimes y),$$

or by the part of the theorem already proved, $0 \otimes 0 = 0 \oplus 0 = 0$.

We now restate (D) in modified form as:

THEOREM IV. *If a, b, c belong to K, then* $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.

Theorems I–IV exhibit the properties of a field in the ordinary sense.

3. Consistency and Independence of the Postulates. We consider the three postulate-sets:

$$\Delta_n\text{: } (A1),\ (A2),\ (M1),\ (M2),\ (D),\ (N_n)\quad (n > 1).$$

$$\Delta'\text{: } (A1),\ (A2),\ (M1),\ (M2),\ (D),\ (N').$$

$$\Delta''\text{: } (A1),\ (A2),\ (M1),\ (M2),\ (D),\ (N'').$$

As the first five postulates alone are in each case necessary and sufficient for the field properties, we need only ask whether the adjunction of the last postulate permits the existence of a field. E. H. Moore has shown* that every finite field is abstractly equivalent to a Galois field; since the number of elements of a Galois field is always a power of a prime and for every power of a prime one and only one Galois field exists, *the set* Δ_n *is consistent when and only when* n *is a power of a prime; it is then categorical.*

Examples of fields of countably and continuously infinite numbers of elements are furnished by the classes of all rational numbers and all real numbers respectively, with

$$a \oplus b = a + b, \qquad a \otimes b = ab.$$

Hence *each of the sets* Δ', Δ'' *is consistent.*†

The consistency of each Δ-set carries with it the independence of the N-postulate in each of the other Δ-sets. The independence of each remaining postulate is shown, simultaneously for the three Δ-sets, in the usual way by the exhibition of systems satisfying all the postulates except the one considered. Agreeing throughout to take for K in Δ_n the n distinct integers modulo n, in Δ' all rational numbers, and in Δ'' all real numbers, we list the independence-systems as follows:

$[A1]$ $a \oplus b = b;$ $\qquad a \otimes b = a + b.$

$[A2]$ $a \oplus b = a;$ $\qquad a \otimes b = a + b.$

$[M1]$ $a \oplus b = a + b;$ $\qquad a \otimes b = b.$

$[M2]$ $a \oplus b = a + b;$ $\qquad a \otimes b = 0.$

$[D]$ $a \oplus b = a + b;$ $\qquad a \otimes b = a + b.$

Thus *the postulates of each set* Δ_n, Δ', Δ'' *are independent.*

4. Further Considerations on Independence. It will be observed that for each of the independence-systems just given the values of $a \oplus b$ and $a \otimes b$ are defined without exception as elements of K; thus *none of the postulates is deducible from the others even with the aid of the two new postulates*:

($A3$) *If* a *and* b *belong to* K, *then* $a \oplus b$ *belongs to* K.

($M3$) *If* a *and* b *belong to* K, *then* $a \otimes b$ *belongs to* K.

Peculiar interest attaches to the independence-systems for those postulates—($A1$) and ($M1$)—which demand for addition and multiplication our combination-substitute for the associative and commutative laws.

* Mathematical Papers Read at the International Mathematical Congress, Chicago (1893), p. 208.

† Neither is categorical. Possible distinct forms of countably infinite fields are discussed by de Séguier, Théorie des groupes finis (1904), p. 51.

Since $[A1]$ preserves the associative law for addition and $[M1]$ that for multiplication, $(A1)$ *is not deducible from the other postulates even with the aid of* $(A3)$, $(M3)$, *and the associative law for addition*; $(M1)$ *is not deducible from the other postulates even with the aid of* $(A3)$, $(M3)$, *and the associative law for multiplication.*

We are naturally led to ask whether $(A1)$ is a consequence of the other postulates together with $(A3)$, $(M3)$, and the commutative law for addition; and similarly for $(M1)$. For the sets Δ', Δ'' the first question is at once answered in the negative. Taking for K all rational numbers or all real numbers, we have the independence-systems:

$$[A1] \qquad a \oplus b = -a - b; \qquad a \otimes b = ab.$$

We can also construct an independence-system for Δ_n. When n is odd, take for K the n distinct integers modulo n, and define:

$$[A1] \qquad a \oplus b = \frac{n+1}{2}(a + b); \qquad a \otimes b = a + b.$$

When n is even and greater than 4, take for K the $(n - 1)$ distinct integers modulo $(n - 1)$, and a special element z; define:

$$[A1] \qquad a \oplus b = \frac{n}{2}(a + b)$$

unless $a = z$, $b = z$, or $a = b$; $a \oplus z = z \oplus a = a$; $a \oplus a = z$;

$$a \otimes b = a + b$$

unless $a = z$ or $b = z$; $a \otimes z = z \otimes a = z$.

In case $n = 4$, we use the four elements 0, 1, 2, z, with the laws of combination:

$[A1]$

$\oplus$	0	1	2	z
0	z	0	2	1
1	0	z	1	2
2	2	1	z	0
z	1	2	0	z

$\otimes$	0	1	2	z
0	0	1	2	z
1	1	2	0	z
2	2	0	1	z
z	z	z	z	z

For $n = 2$, the associative law is easily deduced from the other postulates. Therefore $(A1)$ *is not deducible from the other postulates of* Δ', Δ'', *or of* Δ_n *if* $n > 2$, *even with the aid of* $(A3)$, $(M3)$, *and the commutative law for addition.*

Let us now study the corresponding question for $(M1)$. With the set Δ', choose for K all ordinary complex numbers with rational components; with Δ'', all complex numbers*; we have the independence-definitions:

$$[M1] \qquad a \oplus b = a + b; \qquad a \otimes b = \bar{a}\bar{b}.$$

---

* As usual, we denote by $\bar{a}$ the conjugate complex number to a.

The results for Δ_n are less simple. Let us assume for a class K all the postulates of Δ_n except $(M1)$, and also $(A3)$, $(M3)$, and the commutative law for multiplication. By Theorem I, K is an abelian group with respect to $\oplus$. We agree to write

$$a \oplus a \oplus \cdots \oplus a \ (\nu \text{ terms}) = \nu a,$$

ν being any positive integer; we have the identities:

$$\nu(a \oplus b) = \nu a \oplus \nu b,$$

and (by continued application of (D))

$$\nu(a \otimes b) = \nu a \otimes b.$$

For any element $a \neq 0$ there are positive integers π such that

$$(\pi + 1)a = a;$$

the least such positive integer π_0 is the period of a, and any other such positive integer π is a multiple of π_0. Let π_0, κ_0 be the periods of the elements a, b respectively. By $(M2)$ choose x, y so that

$$a \otimes x = b, \qquad b \otimes y = a.$$

Then

$$(\pi_0 + 1)b = (\pi_0 + 1)(a \otimes x) = (\pi_0 + 1)a \otimes x = a \otimes x = b,$$

$$(\kappa_0 + 1)a = (\kappa_0 + 1)(b \otimes y) = (\kappa_0 + 1)b \otimes y = b \otimes y = a;$$

thus π_0 is a multiple of κ_0 and κ_0 a multiple of π_0, so that $\pi_0 = \kappa_0$. Hence all elements $\neq 0$ have the same period, and by a well-known theorem of group-theory K is an abelian group, with respect to $\oplus$, of order p^μ and type $(1, 1, \cdots, 1)$.

If we choose μ independent generators $u_1, u_2, \cdots, u_\mu$ of the group, any element of K is expressible in the form $\nu_1 u_1 \oplus \nu_2 u_2 \oplus \cdots \oplus \nu_\mu u_\mu$, where the coefficients $\nu_1, \nu_2, \cdots \nu_\mu$ are integers modulo p; we have to deal with a finite linear commutative algebra, in which division by a non zero element is uniquely possible.* In case $\mu = 1$, every element is of the form νu; furthermore

$$\nu u \otimes \nu' u = \nu(u \otimes \nu' u) = \nu(\nu' u \otimes u) = \nu\nu'(u \otimes u);$$

thus the associative law is satisfied, since

$$\nu u \otimes (\nu' u \otimes \nu'' u) = \nu\nu'\nu''[u \otimes (u \otimes u)] = \nu\nu'\nu''[(u \otimes u) \otimes u]$$
$$= (\nu u \otimes \nu' u) \otimes \nu'' u.$$

* *Possible*, on account of $(M2)$; *uniquely* so, since the form $a \otimes x$ represents an element of K, by $(M3)$, and must run through all elements of K as x runs through all elements of K, by $(M2)$.

We show finally that if $\mu > 1$, it is possible to find a system in which the associative law fails. To this end take for K the Galois field $GF[p^\mu]$; define:

$$[M1] \qquad a \oplus b = a + b; \qquad a \otimes b = a^p b^p.$$

All the laws of addition evidently hold, likewise ($M3$) and the commutative law for multiplication. (D) is also satisfied in view of the identity in the field:

$$(a + b)^p = a^p + b^p.$$

Lastly, ($M2$) is satisfied by choosing

$$x = a^{-1} b^{p^{\mu-1}}.$$

The associative law fails unless a, b, or $c = 0$, or $a^{p-1} = c^{p-1}$,—surely, then, if $a = b = 1$, $c =$ a primitive root in the field.*

We collect these results as follows: ($M1$) *is not deducible from the other postulates of* Δ', Δ'', *or of* Δ_n *if* n *is a power higher than the first of a prime, even with the aid of* ($A3$), ($M3$), *and the commutative law for multiplication*; ($M1$) *is deducible from these postulates if* n *is a prime*; *if* n *is not a power of a prime, the postulates of* Δ_n *without* ($M1$), *but with* ($A3$), ($M3$), *and the commutative law for multiplication, are inconsistent.*

Cornell University,
February, 1913.

* In the finite algebras thus defined there is no idemfacient element; hence these algebras are not of the type studied by Dickson, Transactions of the American Mathematical Society, vol. 7 (1906), p. 370 and p. 514.

ON CONTINUED FRACTIONS IN NON-COMMUTATIVE QUANTITIES.

By J. H. M. Wedderburn.

The object of this note is to investigate the properties of simple continued fractions when the terms are not necessarily commutative with one another. The terms, for instance, may be the product of a function of x and differential operator D_x; or they may be matrices; or in fact any functional operator or hypercomplex quantity for which addition obeys the ordinary laws of algebra and multiplication is associative and distributive. In the demonstrations it is always assumed that an inverse exists, but, as the relations given are all integral identities, the truth of the theorems does not depend on the existence of an inverse: the use of the inverse could therefore no doubt have been avoided, as by studying Euclid's algorithm in place of continued fractions for instance, but it does not seem that there would be any material gain in doing so.

1. Consider the continued fraction

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\cdots.$$

When we come to form the convergents of this expression, it is immediately obvious that there are two principal ways of writing them according as we put the denominator of the convergent on the right or on the left of numerator. For instance,

$$a_1 + \frac{1}{a_2} = (a_1a_2 + 1)a_2^{-1} = a_2^{-1}(a_2a_1 + 1),$$

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3} = (a_1a_2a_3 + a_1 + a_3)(a_2a_3 + 1)^{-1},$$

$$= (a_3a_2 + 1)^{-1}(a_3a_2a_1 + a_1 + a_3),$$

and so on.

When the denominator is on the right, the nth convergent will be denoted by $p_nq_n^{-1}$, and when the denominator is on the left by $\bar{q}_n^{-1}\bar{p}_n$.

Theorem 1.

$$p_n = p_{n-1}a_n + p_{n-2}, \qquad q_n = q_{n-1}a_n + q_{n-2}$$

$$\bar{p}_n = a_n\bar{p}_{n-1} + \bar{p}_{n-2}, \qquad \bar{q}_n = a_n\bar{q}_{n-1} + \bar{q}_{n-2}$$

The proof is exactly the same as in ordinary algebra.

Assume for a particular value of n that

$$p_n q_n^{-1} = (p_{n-1}a_n + p_{n-2})(q_{n-1}a_n + q_{n-2})^{-1},$$

then

$$\begin{aligned} p_{n+1}q_{n+1}^{-1} &= [p_{n-1}(a_n + a_{n+1}^{-1}) + p_{n-2}][q_{n-1}(a_n + a_{n+1}^{-1}) + q_{n-2}]^{-1} \\ &= [(p_{n-1}a_n + p_{n-2})a_{n+1} + p_{n-1}]a_{n+1}^{-1}[\{(q_{n-1}a_n + q_{n-2})a_{n+1} \\ &\qquad\qquad + q_{n-1}\}a_{n+1}^{-1}]^{-1} \\ &= (p_n a_{n+1} + p_{n-1})(q_n a_{n+1} + q_{n-1})^{-1}. \end{aligned}$$

But the theorem is readily verified for $n = 1, 2$ and hence it is true for all values. The second part of the theorem follows in exactly the same way.

THEOREM 2. $$\bar{q}_n p_n - \bar{p}_n q_n = 0.$$

For

$$p_n q_n^{-1} = \bar{q}_n^{-1}\bar{p}_n$$

since each is equal to the value of the nth convergent.

THEOREM 3.
$$\begin{aligned} (i)\quad & \bar{q}_{n-1}p_n - \bar{p}_{n-1}q_n = (-1)^n, \\ (ii)\quad & \bar{q}_n p_{n-1} - \bar{p}_n q_{n-1} = (-1)^{n+1}, \\ (iii)\quad & p_n\bar{q}_{n-1} - p_{n-1}\bar{q}_n = (-1)^n, \\ (iv)\quad & q_n\bar{p}_{n-1} - q_{n-1}\bar{p}_n = (-1)^{n+1}. \end{aligned}$$

Let

$$\Delta_n = \bar{q}_{n-1}p_n - \bar{p}_{n-1}q_n, \qquad \bar{\Delta}_n = \bar{q}_n p_{n-1} - \bar{p}_n q_{n-1}.$$

By Theorem 1

$$\begin{aligned} \Delta_n &= \bar{q}_{n-1}(p_{n-1}a_n + p_{n-2}) - \bar{p}_{n-1}(q_{n-1}a_n + q_{n-2}) \\ &= (\bar{q}_{n-1}p_{n-1} - \bar{p}_{n-1}q_{n-1})a_n + \bar{q}_{n-1}p_{n-2} - \bar{p}_{n-1}q_{n-2} \\ &= \bar{q}_{n-1}p_{n-2} - \bar{p}_{n-1}q_{n-2} = \bar{\Delta}_{n-1}, \end{aligned}$$

since, by Theorem 2,

$$\bar{q}_{n-1}p_{n-1} - \bar{p}_{n-1}q_{n-1} = 0.$$

Similarly $\bar{\Delta}_n = \Delta_{n-1}$. Hence Δ_n equals Δ_2 if n is even, and $\bar{\Delta}_2$ if n is odd. But

$$\Delta_2 = a_1a_2 + 1 - a_1a_2 = 1, \qquad \bar{\Delta}_2 = a_2a_1 - (a_2a_1 + 1) = -1.$$

Hence $\Delta_n = (-1)^n$: similarly $\bar{\Delta}_n = (-1)^{n+1}$.

Again if

$$\Delta_n' = p_n\bar{q}_{n-1} - p_{n-1}\bar{q}_n,$$

then

$$\begin{aligned} \Delta_n' &= (p_{n-1}a_n + p_{n-2})\bar{q}_{n-1} - p_{n-1}(a_n\bar{q}_{n-1} + \bar{q}_{n-2}) \\ &= p_{n-2}\bar{q}_{n-1} - p_{n-1}\bar{q}_{n-2} = -\Delta_{n-1}' \end{aligned}$$

and $\Delta_2' = 1$: hence $\Delta_n' = (-1)^n$.

(iv) is proved in exactly the same fashion.

Since $p_n, q_n, \bar{p}_n, \bar{q}_n$ are integral when $a_1, a_2, \cdots$ are, it follows from these relations that p_n and q_n or $\bar{p}_n$ and $\bar{q}_n$ have in general no common factor, although owing to the indefinite nature of the a's no definite conclusion can be drawn. If for example they are differential operators mentioned in the introduction, then the result is valid. For if p_n and q_n have a right-handed factor in common, say

$$p_n = p_n'r, \qquad q_n = q_n'r,$$

then, if y is the dependent variable, the equations $p_ny = 0$ and $q_ny = 0$ have in common the solutions of $ry = 0$: but if y_1 were such a solution, then

$$(\bar{q}_{n-1}p_n - \bar{p}_{n-1}q_n)y_1 = 0,$$

which, by the foregoing theorem, is only possible if $y_1 = 0$.

THEOREM 4. $p_n\bar{p}_{n-1} = p_{n-1}\bar{p}_n, \quad q_n\bar{q}_{n-1} = q_{n-1}\bar{q}_n.$

For

$$\begin{aligned}(p_n\bar{p}_{n-1})^{-1} &= \bar{p}_{n-1}^{-1}p_n^{-1} = (-1)^n\bar{p}_{n-1}^{-1}(\bar{q}_{n-1}p_n - \bar{p}_{n-1}q_n)p_n^{-1} \\ &= (-1)^n[\bar{p}_{n-1}^{-1}\bar{q}_{n-1} - q_np_n^{-1}] \\ &= (-1)^n[q_{n-1}p_{n-1}^{-1} - \bar{p}_n^{-1}\bar{q}_n] \\ &= (-1)^n\bar{p}_n^{-1}[\bar{p}_nq_{n-1} - \bar{q}_np_{n-1}]p_{n-1}^{-1} \\ &= \bar{p}_n^{-1}p_{n-1}^{-1} = (p_{n-1}\bar{p}_n)^{-1}.\end{aligned}$$

Similarly

$$q_n\bar{q}_{n-1} = q_{n-1}\bar{q}_n.$$

Finally, if C_n is the nth convergent, then as in ordinary algebra we have

THEOREM 5. $C_n - C_{n-2} = (-1)^n\bar{q}_{n-2}^{-1}a_nq_n^{-1} = (-1)^n\bar{q}_n^{-1}a_nq_{n-2}^{-1}$.

For

$$\begin{aligned}C_n - C_{n-2} &= \bar{q}_{n-2}^{-1}(\bar{q}_{n-2}p_n - \bar{p}_{n-2}q_n)q_n^{-1} \\ &= (\bar{q}_{n-2}^{-1}[\bar{q}_{n-2}(p_{n-1}a_n + p_{n-2}) - \bar{p}_{n-2}(q_{n-1}a_n + q_{n-2})]q_n^{-1} \\ &= \bar{q}_{n-2}^{-1}[(\bar{q}_{n-2}p_{n-1} - \bar{p}_{n-2}q_{n-1})a_n + \bar{q}_{n-2}p_{n-2} - \bar{p}_{n-2}q_{n-2}]q_n^{-1} \\ &= (-1)^n\bar{q}_{n-2}^{-1}a_nq_n^{-1},\end{aligned}$$

and similarly

$$C_n - C_{n-2} = (-1)^n\bar{q}_n^{-1}a_nq_{n-2}^{-1}$$

also, from Theorem 3(i)

$$C_n - C_{n-1} = p_nq_n^{-1} - \bar{q}_{n-1}^{-1}p_{n-1} = (-1)^n\bar{q}_{n-1}^{-1}q_n^{-1}.$$

2. So far no special restriction has been imposed on the distributive

operators $a_1, a_2, \cdots$. We shall now assume that they are linear differential operators of the form $\Sigma r_n D^n$ where $D = d/dx$ and r_n are functions of x which possess derivatives so far as they are required in the processes employed below.

If p and q are two such operators, the degree of q being not higher than that of p, then we can by the process of ordinary division find operators $a, b, \bar{a}, \bar{b}$ such that

$$p = qa + b, \qquad p = \bar{a}q + \bar{b}.$$

The formal proof* of this is left to the reader. If therefore p and q have no factor in common on the right, we can express pq^{-1} uniquely as a simple terminating continued fraction of the kind we have considered above. If they have a common factor, this process of continued division determines the H.C.F. just as in the theory of ordinary polynomials.

Consider now the following system of ordinary linear differential equations in two dependent variables,

$$U_1 = py_1 + ry_2 = u_1, \qquad U_2 = qy_1 + sy_2 = u_2, \tag{1}$$

p, q, r and s being linear differential operators.

Let the H.C.F. of p and q be c and let $p = p_n c$, $q = q_n c$ where

$$pq^{-1} = p_n q_n^{-1} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n}.$$

Multiplying the first equation by $\bar{q}_n$ and the second by $\bar{p}_n$ and subtracting, and then performing the same operation with $\bar{q}_{n-1}$ and $\bar{p}_{n-1}$ in place of $\bar{q}_n$ and $\bar{p}_n$ and using the result obtained in Theorems 2 and 3, viz.,

$$\bar{q}_n p_n = \bar{p}_n q_n, \qquad \bar{q}_{n-1} p_n - \bar{p}_{n-1} q_n = (-1)^n,$$

we obtain a new system

$$\begin{aligned} V_1 &\equiv \bar{q}_n U_1 - \bar{p}_n U_2 = (\bar{q}_n r - \bar{p}_n s) y_2 = \bar{q}_n u_1 - \bar{p}_n u_2 \equiv v_1, \\ V_2 &\equiv \bar{q}_{n-1} U_1 - \bar{p}_{n-1} U_2 = (-1)^n c y_1 + (\bar{q}_{n-1} r - \bar{p}_{n-1} s) y_2 = \bar{q}_{n-1} u_1 - \bar{p}_{n-1} u_2 \equiv v_2 \end{aligned} \tag{2}$$

and every solution of the original equation is a solution of (2) also.

Conversely

$$\begin{aligned} -p_{n-1} V_1 + p_n V_2 &= (p_n \bar{q}_{n-1} - p_{n-1} \bar{q}_n) U_1 + (p_{n-1} \bar{p}_n - p_n \bar{p}_{n-1}) U_2 = (-1)^{n+1} U_1, \\ q_{n-1} V_1 - q_n V_2 &= (q_{n-1} \bar{q}_n - q_n \bar{q}_{n-1}) U_1 + (q_n \bar{p}_{n-1} - q_{n-1} \bar{p}_n) U_2 = (-1)^{n+1} U_2, \end{aligned}$$

and exactly the same relations hold between v_1, v_2 and u_1, u_2. Hence every solution of the second system is also a solution of the first, i. e., the two are equivalent.

* See Pincherle, Le operazioni distributive, Bologna, 1901, p. 263.

Now (2) has the form

$$ay_2 = v_1,$$
$$(-1)^n cy_1 + by_2 = v_2,$$

hence we have reduced the solution of the original system to the solving of a chain of equations in one variable. The same method can evidently be used to reduce a system in any number of dependent variables to a sequence of equations in one variable just as in the theory of systems with constant coefficients.*

PRINCETON, 1911.

* Cf. Chrystal, Trans. R. S. E., vol. 38 (1895), p. 163.

A NEW TYPE OF SOLUTION OF MAXWELL'S EQUATIONS.

By H. Bateman.

1. There is some experimental evidence of the existence of radiations which are apparently corpuscular in character but also possess the properties of waves or pulses travelling through a continuous medium. The observed phenomena can probably be explained satisfactorily by supposing that the effective part of the disturbance is concentrated in certain regions and almost absent in others. It may, however, be of some interest to inquire if there are any solutions of Maxwell's equations which correspond to a corpuscular type of radiation. The solution which is discussed here fulfils this requirement but is of a type which is generally rejected in the treatment of the boundary problems of mathematical physics. It may not, therefore, correspond to any physical reality. Nevertheless, I have thought it worth while to place the solution on record as it furnishes a good example of the theory of the mutual connection of electromagnetic fields and transformations which can be applied to them. A theory which I have developed in some previous papers.

2. Consider a point M moving along a curve Γ with a velocity v less than c the ve'ocity of light. We may represent its position at time τ by the equations

(1) $$x = \xi(\tau), \qquad y = \eta(\tau), \qquad z = \zeta(\tau).$$

Associated with any space-time point (x, y, z, t) there is just *one** time τ such that if $t \geq \tau$

(2) $$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2(t - \tau)^2.$$

The corresponding position of M is called its *effective position* relative to (x, y, z, t).

Now let $l(\tau)$, $m(\tau)$, $n(\tau)$, $p(\tau)$ be functions connected by the equations

(3) $$l(\tau)\xi'(\tau) + m(\tau)\eta'(\tau) + n(\tau)\zeta'(\tau) = c^2p(\tau), \qquad l^2 + m^2 + n^2 = c^2p^2,$$

and write

(4) $$\sigma = l(x - \xi) + m(y - \eta) + n(z - \zeta) - c^2p(t - \tau).$$

It is easy to verify by a slight modification of the method used in a former

* A. Liénard, L'éclairage électrique, vol. 16 (1898), pp. 5, 53, 106; A. W. Conway, Proc. London Math. Soc., Ser. 2, vol. 1, p. 154 (1903); H. Minkowski, Raum und Zeit, Phys. Zeitschr. (1909); H. Bateman, Manchester Memoirs (1910). See also G. A. Schott's Adams Prize Essay, 'Electromagnetic Radiation,' Cambr. Univ. Press (1912).

paper* that the function

$$V = \frac{f(\tau)}{\sigma}$$

is a solution of the wave equation

$$\Omega(v) \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0.$$

Moreover, if we derive an electromagnetic field from the potentials

$$(5) \qquad A_x = \frac{l}{\sigma}, \qquad A_y = \frac{m}{\sigma}, \qquad A_z = \frac{n}{\sigma}, \qquad \Phi = \frac{p}{\sigma},$$

we find that the equation

$$\text{div } A + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0.$$

is satisfied, and that there is no charge or convection current at space-time points for which $\sigma \neq 0$.

The quantity σ vanishes when (x, y, z, t) is on Γ and also when it lies on one of two lines through the effective position of M. To see this let (ξ', η', ζ') be regarded as the coördinates of a point U within the sphere

$$X^2 + Y^2 + Z^2 = c^2$$

which we shall denote by S. The conditions (3) imply that the point Q whose coördinates are $(l/cp, m/cp, n/cp)$ lies on the polar plane of U and also on the sphere. Now the polar plane of U lies outside the sphere and so only meets it in imaginary points, hence the ratios l/p, m/p, n/p are not all real. Since, moreover, the polar plane of U meets the sphere S in an imaginary circle C it is clear that there are ∞' possible sets of values of l, m, n, p. Now let $(x - \xi)/c(t - \tau)$, $(y - \eta)/c(t - \tau)$, $(z - \zeta)/c(t - \tau)$ be regarded as the coördinates of a point P on the sphere S. The tangent plane at P meets the circle C in two imaginary points R, R'. If one of these coincides with the point Q the expression σ vanishes.

Now if $Q \equiv R$ the coördinates of R' can be written down at once by changing i into $-i$ in the coördinates of R. Having found R' we can determine P by drawing tangent planes through RR' to the sphere. The points of contact of these two planes will be the two possible positions of P. Now these points lie on the polar line of RR' and this is a line through U perpendicular to RR' which cuts the sphere in two *real* points PP'.

Taking the real (or i times the imaginary) parts of the potentials (5) we obtain a real electromagnetic field with the following properties. *There is a primary singularity M moving with a velocity less than that of light along*

* *Annals of Mathematics*, 2d series, vol. 14, Dec. (1912).

a curve Γ. There are secondary singularities radiating with the velocity of light from points of the curve Γ, there being two rectilinear rays through each point of this curve. The direction of motion of M bisects the angle between the two rays and the angle which it makes with either has a cosine equal to v/c, where v is the velocity of M.

It should be noticed that if M is moving along a straight line with constant velocity and l, m, n, p are regarded as constants, the electric and magnetic forces derived from the potentials (5) are everywhere null. *Hence the electromagnetic field under consideration is of a type which only comes into existence when the velocity of the primary singularity changes,* or the directions of the rays alter.

The components of the electric and magnetic vectors D, H are given by equations of type

$$H_x \equiv \frac{\partial}{\partial y}\left(\frac{n}{\sigma}\right) - \frac{\partial}{\partial z}\left(\frac{m}{\sigma}\right) = \left(\frac{n'}{\sigma} - \frac{Kn}{\sigma^2}\right)\frac{\partial \tau}{\partial y} - \left(\frac{m'}{\sigma} - \frac{Km}{\sigma^2}\right)\frac{\partial \tau}{dz},$$

$$D_x \equiv -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{l}{\sigma}\right) - \frac{\partial}{\partial x}\left(\frac{cp}{\sigma}\right) = -\frac{1}{c}\left(\frac{l'}{\sigma} - \frac{Kl}{\sigma^2}\right)\frac{\partial \tau}{\partial t} - c\left(\frac{p'}{\sigma} - \frac{Kp}{\sigma^2}\right)\frac{\partial \tau}{\partial x},$$

where

$$K \equiv l'(x - \xi) + m'(y - \eta) + n'(z - \zeta) - c^2 p'(t - \tau).$$

It follows from these equations that*

$$\frac{\partial \tau}{\partial x}H_x + \frac{\partial \tau}{\partial y}H_y + \frac{\partial \tau}{\partial z}H_z = 0,$$

$$\frac{1}{c}\frac{\partial \tau}{\partial t}H_x + \frac{\partial \tau}{\partial y}D_z - \frac{\partial \tau}{\partial z}D_y = 0,$$

(6)

$$\frac{\partial \tau}{\partial x}D_x + \frac{\partial \tau}{\partial y}D_y + \frac{\partial \tau}{\partial z}D_z = 0,$$

$$\frac{1}{c}\frac{\partial \tau}{\partial t}D_x - \frac{\partial \tau}{\partial y}H_z + \frac{\partial \tau}{\partial z}H_y = 0.$$

Hence the electric and magnetic vectors are at right angles and are equal in magnitude. Poynting's vector has components proportional to $\partial\tau/\partial x$, $\partial\tau/\partial y$, $\partial\tau/\partial z$ and so is along the radius from the effective position of M.

When the real or imaginary parts of D and H only are retained all the above conclusions remain valid.

3. Another conclusion which may be drawn from equations (6) is that the electromagnetic field can be transformed into another one of the same

* We make use here of the relations

$$\frac{\frac{\partial \tau}{\partial x}}{x - \xi} = \frac{\frac{\partial \tau}{\partial y}}{y - \eta} = \frac{\frac{\partial \tau}{\partial z}}{z - \zeta} = -\frac{\frac{\partial \tau}{\partial t}}{c^2(t - \tau)}.$$

general type but associated with a different curve Γ'. The equations of the transformation are*

$$(7)\qquad \begin{aligned} x' &= xf(\tau) - \int_{\tau_0}^{\tau} f'(\tau)\xi(\tau)d\tau, & z' &= zf(\tau) - \int_{\tau_0}^{\tau} f'(\tau)\zeta(\tau)d\tau, \\ y' &= yf(\tau) - \int_{\tau_0}^{\tau} f'(\tau)\eta(\tau)d\tau, & t' &= tf(\tau) - \int_{\tau_0}^{\tau} \tau f'(\tau)d\tau, \end{aligned}$$

where τ is defined in terms of x, y, z, t by the equations (2) and τ_0 is a constant.

This transformation makes the curve Γ correspond to a curve Γ' defined by the equations

$$(8)\qquad x' = \alpha(\tau), \qquad y' = \beta(\tau), \qquad z' = \gamma(\tau), \qquad t' = \theta(\tau),$$

where

$$\begin{aligned} \alpha'(\tau) &= f(\tau)\xi'(\tau), & \beta'(\tau) &= f(\tau)\eta'(\tau), \\ \gamma'(\tau) &= f(\tau)\zeta'(\tau), & \theta'(\tau) &= f(\tau). \end{aligned}$$

It is clear from these equations that the two curves Γ and Γ' have the same spherical indicatrix and that the velocities at corresponding points are equal. If Γ is a plane curve so also is Γ'. The relation

$$(9)\qquad (x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 = c^2(t' - \theta)^2$$

is evidently a consequence of (2). The inequality $t' \geq \theta$ is a consequence of $t \geq \tau$ if $f(\tau)$ is positive and then θ increases with τ.

Defining the potentials for the corresponding electromagnetic field by the equations

$$(10)\qquad A_x' = \frac{l}{\sigma'}, \qquad A_y' = \frac{m}{\sigma'}, \qquad A_z' = \frac{n}{\sigma'}, \qquad \Phi = \frac{cp}{\sigma'},$$

where

$$\sigma' = l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma) - c^2p(t' - \theta),$$

it is easy to see that $\sigma' = \sigma f(\tau)$ and

$$(11)\qquad A_x'dx' + A_y'dy' + A_z'dz' - c\Phi'dt' = A_xdx + A_ydy + A_zdz - c\Phi dt + \frac{f'}{f}d\tau.$$

From this equation we may deduce that†

$$(12)\qquad \begin{aligned} &H_x'd(y', z') + H_y'd(z', x') + H_z'd(x', y') + cD_x'd(x', t') \\ &\qquad + cD_y'd(y', t') + cD_z'd(z', t') = H_xd(y, z) + H_yd(z, x) \\ &\qquad + H_zd(x, y) + cD_xd(x, t) + cD_yd(y, t) + cD_zd(z, t), \end{aligned}$$

and so the relations between the electric and magnetic vectors at corre-

* Proc. London Math. Soc., Ser. 2, vol. 10 (1911), p. 96.

† The method of derivation is explained in former papers. See the next reference.

sponding points of the two corresponding fields may be written down at once.

The fact that this transformation implies the covariance of the electromagnetic equations is a consequence of equations (6) and (12) as is indicated by the analysis in a former paper.*

4. In conclusion it may be worth while to point out that the solution of Laplace's equation given in my former paper may be obtained by contour integration.

It is known that $z/\sqrt{x+iy}\cdot 1/r^2$ is a solution of Laplace's equation: for it may be written in the form

$$\frac{1}{2\sqrt{x+iy}}\left[\frac{1}{z+i\rho}+\frac{1}{z-i\rho}\right],$$

where $\rho^2 = x^2 + y^2$, and $(1/\sqrt{x+iy})f(z+i\rho)$ is known to be a solution.

Generalizing this by a change of rectangular axes we may say that

$$\frac{\xi'(u)[x-\xi]+\eta'(u)[y-\eta]+\zeta'(u)[z-\zeta]}{\{l(u)[x-\xi]+m(u)[y-\eta]+n(u)[z-\zeta]\}^{\frac{1}{2}}}\,\frac{1}{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$$

is a solution provided

$$l\xi'(u)+m\eta'(u)+n\zeta'(u)=0,$$
$$l^2+m^2+n^2=0.$$

Integrating round a closed contour containing only one root of the equation

$$[x-\xi(u)]^2+[y-\eta(u)]^2+[z-\zeta(u)]^2=0$$

and no root of the equation

$$l(x-\xi)+m(y-\eta)+n(z-\zeta)=0$$

we obtain the required result.

Another theorem of a similar character is that if $F(X, Y, Z)$ is a homogeneous function of degree $\frac{1}{2}$ satisfying Laplace's equation, then

$$V = F[x-\xi, y-\eta, z-\zeta]$$

is also a solution of Laplace's equation. The corresponding theorems for the equation of wave motion are as follows:

If τ is defined by equation (2) and $F(X, Y, Z, T)$ is a homogeneous function of degree zero satisfying the equation of wave motion, the function

$$V = F[x-\xi, y-\eta, z-\zeta, t-\tau]$$

also satisfies the equation of wave motion.†

* Proc. London Math. Soc., Ser. 2, vol. 8, pp. 223, 469.

† Three particular cases of this result are given by E. T. Whittaker, Proc. London Math. Soc., Ser. 2, vol. 1 (1903).

If $F[X, Y, Z, T, \tau]$ is of degree -1 in X, Y, Z, T and is a homogeneous function of these variables satisfying the equation of wave motion and the additional equation

$$\xi' \frac{\partial F}{\partial X} + \eta' \frac{\partial F}{\partial Y} + \zeta' \frac{\partial F}{\partial Z} - \frac{1}{c^2} \frac{\partial F}{\partial T} = 0,$$

then the funct on

$$V = F[x - \xi(\tau),\ y - \eta(\tau),\ z - \zeta(\tau),\ t - \tau,\ \tau]$$

satisfies the equation of wave motion.

The transformation (7) converts these wave potentials into wave potentials in the (x', y', z', t') coördinates. The new potentials are of the same general character as before but are associated with the curve Γ'.

JOHNS HOPKINS UNIVERSITY,
March 15, 1913.

RELATION BETWEEN THE ZEROS OF A RATIONAL INTEGRAL FUNCTION AND ITS DERIVATE.

By Tsuruichi Hayashi.

The theorem: If $f'(z)$ is the first derivate of a rational integral function $f(z)$, then on the Gaussian plane of complex numbers all the roots of the equation $f'(z) = 0$ lie inside the smallest convex rectilinear polygon surrounding all the roots of the equation $f(z) = 0$, is well known.* All the proofs for this algebraical theorem are given by using dynamical conceptions, as far as I know.† So I shall prove it here without using any idea of force, but still by starting from the equation

$$(1) \qquad \frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} = 0,$$

in which $z_1, z_2, \cdots, z_n$ are the roots of $f(z) = 0$. Let a be any complex number and let equation (1) take the form

$$\sum_{r=1}^{n} \frac{1}{1 - \dfrac{z_r - a}{z - a}} = 0.$$

Put

$$z - a = \xi + \eta i, \qquad z_r - a = \xi_r + \eta_r i,$$

then

$$(2) \qquad \sum_{r=1}^{n} \frac{1}{1 - \dfrac{z_r - a}{z - a}} = \sum_{r=1}^{n} \frac{\left(1 - \dfrac{\xi\xi_r + \eta\eta_r}{\xi^2 + \eta^2}\right) + \dfrac{\xi\eta_r - \eta\xi_r}{\xi^2 + \eta^2} i}{\left(1 - \dfrac{\xi\xi_r + \eta\eta_r}{\xi^2 + \eta^2}\right)^2 + \left(\dfrac{\xi\eta_r - \eta\xi_r}{\xi^2 + \eta^2}\right)^2} = 0.$$

If we observe the imaginary part only of this equation, we find that ξ, η can not take those values for which all the expressions $\xi\eta_r - \eta\xi_r$ have the same sign,‡ since, if this be so, the imaginary part does not vanish. Now the equations

$$\xi\eta_r - \eta\xi_r = 0 \qquad (r = 1, 2, \cdots, n)$$

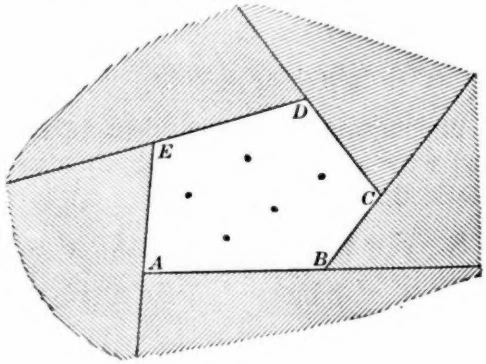
represent the straight lines joining the point a to the points $z_1, z_2, \cdots, z_n$.

* For the literature of this theorem consult L. Fejér's paper "Ueber die Würzel vom kleinsten absoluten Betrage einer algebraischen Gleichung," Math. Annalen, Bd. 65, S. 417, 1908, E. Cesàros' paper, Nouv. Annales de Math., (3)4, p. 329, 1885, and Cesàro-Kowalewski's "Elementares Lehrbuch der algebraischen Analysis," S. 434, 1904.

† See L. Fejér's paper, W. F. Osgood's "Lehrbuch der Funktionentheorie," Bd. I, S. 176, 1907, etc.

‡ Some of these expressions may be zero.

Hence if we make the point coincide with one of the vertices, A say, of the convex rectilinear polygon under consideration, then we find that the point (ξ, η), and therefore the root z of $f'(z) = 0$, can not lie within the exterior angles at A, since these regions are on the same-sign sides of all these straight lines. Next let the point a come to the vertex B; then the root of $f'(z) = 0$ can not lie within the exterior angles at B. Thus by letting the point a



coincide with the vertices of the polygon successively, we find that all the roots of $f'(z) = 0$ can not lie within the shaded portion as in the accompaning diagram. If we consider the real part of equation (2), and the circles

$$1 - \frac{\xi\xi_r + \eta\eta_r}{\xi^2 + \eta^2} = 0$$

whose diameters are the joins of the points a and z_i, we can arrive at the same result.

SENDAI, JAPAN,
January, 1913.

THE INVARIANTS, SEMINVARIANTS AND LINEAR COVARIANTS OF THE BINARY QUARTIC FORM MODULO 2.

By L. E. Dickson.

Denote the quartic form by

$$(1) \qquad ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4,$$

where $a, \cdots, e$ are undetermined integers taken modulo 2 so that $a^2 \equiv a$, etc. Upon replacing x by $x + y$ in (1), we obtain a new quartic with the following coefficients modulo 2:

$$(2) \quad a' = a, \quad b' = b, \quad c' = b + c, \quad d' = b + d, \quad e' = a + b + c + d + e.$$

A polynomial $P(a, \cdots, e)$ with integral coefficients is called a seminvariant of (1) modulo 2 if

$$P(a', \cdots, e') \equiv P(a, \cdots, e) \qquad (\text{mod } 2).$$

Theorem 1. *Every seminvariant is a rational integral function of*

$$(3) \quad a, \quad b, \quad \alpha = (b-1)c, \quad \beta = c + d, \quad \gamma = ac + cd + be,$$
$$\delta = (b-1)(a+c+d-1)e.$$

The functions (3) are easily verified by means to (2) to be seminvariants. The theorem will follow* if we show that these six seminvariants serve to characterize the classes of quartics (1) under the set of two transformations

$$(T) \qquad x \equiv x' + ty', \qquad y \equiv y' \qquad (\text{mod } 2).$$

Two quartics are said to be in the same class if and only if they are transformable into each other by transformations of type (T).

If $b \equiv 1$, we apply (T) for $t = c$ to (1) and get

$$(4) \qquad ax^4 + x^3y + \beta xy^3 + \gamma y^4.$$

If $b \equiv 0$, $a + c + d \equiv 1$, we apply (T) for $t = e$ to (1) and get

$$(5) \qquad ax^4 + \alpha x^2y^2 + (\alpha + \beta)xy^3 \qquad (a + \beta \equiv 1).$$

If $b \equiv 0$, $a + c + d \equiv 0$, (1) becomes

$$(6) \qquad ax^4 + \alpha x^2y^2 + (\alpha + \beta)xy^3 + \delta y^4 \qquad (a + \beta \equiv 0).$$

Since the forms (4)–(6) involve only the seminvariants, the theorem follows.

Theorem 2. *A complete set of linearly independent seminvariants is*

* American Journal of Mathematics, vol. 31 (1909), p. 337.

given by the following twenty:

$$\delta,\ \delta a,\ \delta\alpha,\ \delta a\alpha,\qquad b,\ ba,\ b\beta,\ ba\beta, \tag{7}$$

$$1,\ a,\ \alpha,\ a\alpha,\qquad \beta,\ \beta a,\ \beta\alpha,\ \beta a\alpha,\qquad \gamma,\ \gamma a,\ \gamma\beta,\ \gamma a\beta. \tag{8}$$

The number of forms (4)–(6) is $8 + 4 + 8$, so that there are 20 classes. Hence there are exactly 20 linearly independent seminvariants.* But

$$\delta b \equiv 0,\quad \delta\beta \equiv \delta a,\quad \delta\gamma \equiv \delta\alpha,\quad b\alpha \equiv 0,\quad (b-1)\gamma \equiv \alpha\gamma \equiv \alpha(1+a+\beta). \tag{9}$$

Hence any polynomial in the functions (3) is a linear function of the eight functions (7) and those combinations (8) of a, α, β, γ which lack $\alpha\gamma$. As a check, we may verify that the twenty functions (7), (8) are linearly independent modulo 2.

A seminvariant unaltered by the substitution $(ae)(bd)$, induced by the interchange of x and y, is an invariant. Obvious invariants† are

$$\begin{aligned} L &= b + c + d = b + \beta, \qquad Q = c^2 + bd = \alpha + b\beta,\\ M &= ace(b-1)(d-1) = a\alpha\delta. \end{aligned} \tag{10}$$

From the product of the first two we obtain

$$J = (b-1)(c-1)(d-1) = (L+1)(Q+1) = (\beta+1)(b+\alpha+1). \tag{11}$$

General theorems‡ give this J and

$$I = (a-1)(e-1)J = (a-1)(J + \alpha\delta + \delta). \tag{12}$$

There are two further invariants of the second order:

$$\begin{aligned} V &= ad + be + cL = \alpha + \gamma + a\beta,\\ W &= ae + (a+e)(L+1) = \delta + a + ab + a\gamma + \beta\gamma + a\beta. \end{aligned} \tag{13}$$

From these we get§

$$\begin{aligned} WL &= aeL = a(\delta + \gamma + \beta\gamma),\\ VL + V &= abd + acd + bce + bde = a\alpha + \beta\gamma + ab\beta. \end{aligned} \tag{14}$$

The preceding nine invariants have been expressed linearly in terms of the seminvariants (7), (8). From the general linear function of the latter we subtract constant multiples of

$$I,\ M,\ WL,\ W,\ VL,\ J,\ V,\ Q,\ L,\ 1 \tag{15}$$

---

* Trans. Amer. Math. Soc., vol. 10 (1909), p. 126.

† The algebraic invariants $4I$ and $16J$ of (1) reduce modulo 2 to Q and $V + LQ$. See also end of paper.

‡ Trans. Amer. Math. Soc., vol. 8 (1907), pp. 206–7, § 1, § 3.

§ The eliminant of (1), $x^2 \equiv x$, $y^2 \equiv y$ is aeL. See preceding paper, § 4.

to eliminate in turn the terms in

$$\alpha\delta,\ a\alpha\delta,\ a\delta,\ \delta,\ \beta\gamma,\ \alpha\beta,\ \gamma,\ b\beta,\ \beta,\ 1.$$

It therefore remains to test for invariance a function

$$A = k_1 b + k_2 ba + k_3 ba\beta + k_4 a + k_5 \alpha + k_6 a\alpha + k_7 \beta a + k_8 \beta a\alpha + k_9 \gamma a + k_{10} \gamma a\beta, \tag{16}$$

in which the k's are constants. Applying the substitution $(ae)(bd)$, we get

$$\begin{aligned} A' &= k_1 d + k_2 de + k_3 de(c+b) + k_4 e + k_5(d-1)c + k_6(d-1)ce \\ &\quad + k_7(c+b)e + k_8(b+1)(d-1)ce + k_9\gamma' e + k_{10}\gamma' e(c+b), \\ \gamma' &\equiv ce + bc + ad. \end{aligned} \tag{17}$$

By the terms free of e, $k_i = 0$ $(i = 1, \cdots, 8)$, $k_9 = k_{10}$. Then $k_9 \equiv 0$.

THEOREM 3. *The ten functions* (15) *form a complete set of linearly independent invariants. They are linearly independent of the ten seminvariants in* (16).

The invariants (15) are functions of L, Q, I, V, W since

$$M = I + J + W(Q + L + 1), \qquad J = (L+1)(Q+1). \tag{18}$$

Any polynomial in these five can be expressed as a linear function of the invariants (15) by use of the relations (18) and

$$WV = VL + V + WL, \qquad QV = VL + V + QL, \qquad x^2 \equiv x,\ Ix \equiv 0, \tag{19}$$

where x is any one of the five. By the linear independence of the functions (15) we thus obtain

THEOREM 4. *As a fundamental system of invariants we may take L, Q, I, V, W. No one of these is congruent to a rational integral function of the remaining four.*

A linear covariant of (1) must evidently be of the form

$$C = (i + A)x + (i + A')y,$$

where i is an invariant, while A and A' are given by (16) and (17). This C has the covariant property with respect to the interchange of x and y. Its covariance with respect to $x \equiv x' + y'$, $y \equiv y'$, requires that

$$(i + A)(x + y) + (i + E)y \equiv C,$$

where E is the function derived from A' by the substitution (2). Thus

$$i = A + A' + E.$$

Since A is a seminvariant, this i is evidently unaltered by (2). It will

therefore be an invariant if unaltered by the substitution $(ae)(bd)$. The latter condition is found to give

$$i = k_2(V + L) + k_3(VL + V) + k_4L + k_5Q + k_6(LQ + VL + V) + k_7V + (k_8 + k_9 + k_{10})LQ.$$

Denote by K_i the covariant obtained by setting $k_i \equiv 1$, $k_j \equiv 0 (k \neq i)$. These covariants are linear combinations of products of

$$(20) \qquad \lambda = bx + dy, \qquad \mu = (L + a)x + (L + e)y$$

(viz., K_1, K_4) by invariants. Indeed,

$$K_5 = (Q + \alpha)x + (Q + cd + c)y = b(c + d)x + d(c + b)y = (L + 1)\lambda = Q\lambda,$$

$$K_7 = (V + \beta a)x + (V + ce + be)y = W\lambda + V\mu,$$

$$K_2 + K_7 + \mu = (L + 1)(ax + ey) = (L + 1)\mu = W\mu, \qquad K_3 = (L + 1)K_7,$$

$$K_6 + K_3 = Q\mu = V\lambda + K_7,$$

$$K_8 = c(b + 1)(d + 1)\{(a + 1)x + (e + 1)y\} = QW\mu + K_6 = (VL + V)\lambda + K_3 + K_6,$$

$$K_9 = V(\lambda + \mu) + K_3, \qquad K_{10} = K_9 + WL\lambda.$$

THEOREM 5. *The ten linearly independent linear covariants are functions of the two covariants* (20) *and the invariants.*

In addition to the syzygies between (20) and invariants which are included in the preceding set of relations, we note

$$I\lambda = M\lambda = J\lambda = I\mu = M\mu = 0.$$

The resultants of λ, μ; λ, quartic (1); μ, (1) are respectively $V + L$ $V + LQ$, $L + WL$. The modular invariants of λ and μ are respectively $(b - 1)(d - 1) \equiv Q + L + 1$, $W + L + 1$.

UNIVERSITY OF CHICAGO,
June, 1913.

EXAMPLES OF NORMAL DOMAINS OF RATIONALITY BELONGING TO ELEMENTARY GROUPS.

By G. A. Miller.

Introduction.

The object of the present expository article is to furnish an approach, which is very direct and elementary from a certain point of view, for a study of the theory of groups of normal domains of rationality. Only a few fundamental theorems relating to the theories of groups and domains of rationality will be assumed as known. The most important of these theorems for our purpose may be stated as follows: If the irrational number ρ_1 generates a normal domain and if the other roots $(\rho_2, \rho_3, \cdots, \rho_n)$ of the irreducible equation which ρ_1 satisfies are expressed as rational functions of ρ_1 then we can obtain the group of this domain as a regular substitution group by replacing ρ_1 successively by $\rho_1, \rho_2, \cdots, \rho_n$ in all these rational functions and by noting the permutations of the values of these functions.*

It will always be assumed, in what follows, that the irreducible equation $f(x) = 0$ which ρ_1 satisfies, has rational coefficients and hence it lies in the natural or absolute domain of rationality. Hence, we shall assume that $f(x)$ is not the product of two rational integral functions of x with rational coefficients. In the first section of the present article we shall determine a domain of rationality for each one of the groups whose order is less than 8. Only one of these groups is non-abelian, viz., the symmetric group of order 6. In the second section we shall consider briefly domains belonging to two infinite systems of very elementary groups. The central point of view will be the group and its applications rather than the domain and its properties.

1. Domains Belonging to the Groups whose Orders are less than 8.

In case of the group of order 2 the considerations are very elementary and may appear trivial. For the sake of completeness from the present point of view we shall, however, give some of the details even in this case. The two roots of any quadratic equation $ax^2 + bx + c = 0$ are rational functions of each other since the sum of these roots is $-b/a$. If ρ_1, ρ_2 are these roots it results that $\rho_1 = \varphi_1(\rho_1) = \rho_1$, $\rho_2 = \varphi_2(\rho_1) = -b/a - \rho_1$. By replacing ρ_1 in these functions successively by ρ_1 and ρ_2, and noting the

* Cf. H. Weber, Kleines Lehrbuch der Algebra, 1912, p. 245; F. Cajori, Theory of Equations, 1912, p. 161.

permutations of these values, we obtain the following two substitutions: 1, $(\rho_1\rho_2)$. Hence each root of any irreducible quadratic equation generates a normal domain which belongs to the group of order 2.

In the case of cubic equations it is evident that each root is not necessarily a rational function of every other one since two of these roots may be complex and the third real. It is, however, easy to construct special irreducible cubic equations which have the property that each root is a rational function of some one, and hence to find numbers which generate a normal domain belonging to the group of order 3. To find one such number we may consider the imaginary seventh roots of unity θ, θ^2, θ^3, θ^4, θ^5, θ^6. We shall thus find also a domain belonging to the cyclic group of order 6.

In fact, if we replace θ successively by θ, θ^2, $\cdots$, θ^6 in the following equations:

$$\rho_1 = \theta, \quad \rho_2 = \theta^2, \quad \rho_3 = \theta^3, \quad \rho_4 = \theta^4, \quad \rho_5 = \theta^5, \quad \rho_6 = \theta^6$$

we evidently obtain the group of the totitives mod 7, and this is the cyclic group of order 6.* It therefore results that the number θ generates a domain which belongs to the cyclic group of order 6. Since ρ_1, ρ_6; ρ_2, ρ_5; ρ_3, ρ_4 are the three systems of imprimitivity of this cyclic group, corresponding to its subgroup of order 2, it results that *each of the three numbers*

$$\psi_1 = \theta + \theta^6, \qquad \psi_2 = \theta^2 + \theta^5, \qquad \psi_3 = \theta^3 + \theta^4$$

generates a domain of rationality which belongs to the group of order 3.

To verify this statement we may observe that these three numbers are the roots of the irreducible equation $x^3 + x^2 - 2x + 1 = 0$, and that

$$\psi_1 = \varphi_1(\psi_1) = \psi_1, \quad \psi_2 = \varphi_2(\psi_1) = \psi_1^2 - 2, \quad \psi_3 = \varphi_3(\psi_1) = -\psi_1^2 - \psi_1 + 1,$$
$$\psi_2 = \varphi_1(\psi_2) = \psi_2, \quad \psi_3 = \varphi_2(\psi_2) = \psi_2^2 - 2, \quad \psi_1 = \varphi_3(\psi_2) = -\psi_2^2 - \psi_2 + 1,$$
$$\psi_3 = \varphi_1(\psi_3) = \psi_3, \quad \psi_1 = \varphi_2(\psi_3) = \psi_3^2 - 2, \quad \psi_2 = \varphi_3(\psi_3) = -\psi_3^2 - \psi_3 + 1.$$

Since the given cyclic group of order 6 has for its two systems of imprimitivity, corresponding to its subgroup of order 3, the two sets ρ_1, ρ_2, ρ_4; ρ_3, ρ_5, ρ_6, it results that each of the two numbers $\theta + \theta^2 + \theta^4$, $\theta^3 + \theta^5 + \theta^6$ generates a domain which belongs to the group of order 2. In fact, these numbers are the roots of the irreducible equation $x^2 + x + 2 = 0$.

We proceed to determine a domain for each of the two groups of order 4. If α is an imaginary fifth root of unity it satisfies the irreducible equation $x^4 + x^3 + x^2 + x + 1 = 0$, and the other three roots of this equation are α^2, α^3, α^4. Hence the group of the domain generated by α is the group of the totitives mod 5; that is, the group of the four numbers 1, 2, 3, 4 when these numbers are multiplied together and the products are reduced mod 5.

* Cf. P. Bachmann, Die Elemente der Zahlentheorie, 1892, p. 89.

It is easy to verify, and well known, that this group is the cyclic group of order 5, and hence *each complex fifth root of unity generates a domain belonging to the cyclic group of order* 4.

As an instance of a domain which belongs to the non-cyclic group of order 4 we may mention the one which is generated by $\rho_1 = p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}$, where p_1 and p_2 are distinct rational prime numbers. It is easy to verify that ρ_1 is a root of the irreducible equation

$$x^4 - 2(p_1 + p_2)x^2 + (p_1 - p_2)^2 = 0.$$

Adopting the notation

$$\rho_1 = p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}, \qquad \rho_3 = -p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}},$$
$$\rho_2 = p_1^{\frac{1}{2}} - p_2^{\frac{1}{2}}, \qquad \rho_4 = -p_1^{\frac{1}{2}} - p_2^{\frac{1}{2}},$$

it is clear that the first three powers of ρ_1 give rise to only three linearly independent irrational numbers in the natural domain; viz., $p_1^{\frac{1}{2}}$, $p_2^{\frac{1}{2}}$, and $p_1^{\frac{1}{2}}p_2^{\frac{1}{2}}$. The determinant of the system of the three equations thus formed cannot vanish since there is no rational relation between these powers of ρ_1 because ρ_1 is a root of an irreducible equation of degree 4. Hence these equations can be solved and each of the roots ρ_1, ρ_2, ρ_3, ρ_4 can be expressed as a rational function of any one of them.

It is now easy to see that the group of the domain generated by ρ_1 is actually the non-cyclic group of order 4. In fact, all the substitutions besides the identity of this domain must involve a transposition, since these roots consist of two components, which are linearly independent in the natural domain, and these components differ only with respect to sign. Hence we pass from one of these roots to the other, in the functions which express all of them in terms of a particular one, by means of an operation of period 2. These observations may easily be verified by actually computing the functions in question and then replacing ρ_1 by each of the other roots. The functions are as follows:

$$\rho_1 = \varphi_1(\rho_1) = p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}} = \rho_1,$$

$$\rho_2 = \varphi_2(\rho_1) = p_1^{\frac{1}{2}} - p_2^{\frac{1}{2}} = \frac{\rho_1^3 - 2(p_1 + p_2)\rho_1}{p_2 - p_1},$$

$$\rho_3 = \varphi_3(\rho_1) = -p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}} = \frac{\rho_1^3 - 2(p_1 + p_2)\rho_1}{p_1 - p_2},$$

$$\rho_4 = \varphi_4(\rho_1) = -p_1^{\frac{1}{2}} - p_2^{\frac{1}{2}} = -\rho_1.$$

If ρ_1 is replaced successively by ρ_1, ρ_2, ρ_3, ρ_4, there result the substitutions of the four group; that is, *the number* $p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}$, *where* p_1 *and* p_2 *are distinct rational prime numbers, generates a domain whose group is the non-cyclic group of order* 4.

To find a domain belonging to the group of order 5 we may use the ten complex eleventh roots of unity. Each of these roots generates a domain belonging to the cyclic group of order 10 and the pairs of roots corresponding to the five systems of imprimitivity of this cyclic group must therefore generate a domain belonging to the group of order 5. If these ten complex roots are $\beta, \beta^2, \beta^3, \beta^4, \beta^5, \beta^6, \beta^7, \beta^8, \beta^9, \beta^{10}$, the group of the domain generated by β involves the following substitution:

$$(\beta\beta^2\beta^4\beta^8\beta^5\beta^{10}\beta^9\beta^7\beta^3\beta^6).$$

Hence *each of the five numbers*

$$\beta + \beta^{10}, \quad \beta^2 + \beta^9, \quad \beta^3 + \beta^8, \quad \beta^4 + \beta^7, \quad \beta^5 + \beta^6$$

generates a domain belonging to the group of order 5. It is not difficult to prove that the irreducible equation which has these five numbers as roots is as follows:

$$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1.$$

The domains which have been considered thus far are called *abelian* since they belong to an abelian group. We proceed now to determine a non-abelian domain; viz., one which belongs to the symmetric group of order 6. The domain generated by $\rho_1 = \sqrt{-3} + p^{\frac{1}{3}}$, p being any prime number, is non-abelian. In fact, the six roots of the irreducible equation which is satisfied by ρ_1 are as follows:

$$\rho_1 = \sqrt{-3} + p^{\frac{1}{3}}, \quad \rho_2 = \sqrt{-3} + \omega p^{\frac{1}{3}}, \quad \rho_3 = \sqrt{-3} + \omega^2 p^{\frac{1}{3}},$$

$$\rho_4 = -\sqrt{-3} + p^{\frac{1}{3}}, \quad \rho_5 = -\sqrt{-3} + \omega p^{\frac{1}{3}}, \quad \rho_6 = -\sqrt{-3} + \omega^2 p^{\frac{1}{3}}.$$

The substitution which corresponds to the case when ρ_1 is replaced by ρ_4 merely changes the sign of $\sqrt{-3}$, and hence it is as follows: $(\rho_1\rho_4)(\rho_2\rho_6)(\rho_3\rho_5)$. The substitution corresponding to the case when ρ_2 is replaced by ρ_5 changes the sign of $\sqrt{-3}$ and multiples $p^{\frac{1}{3}}$ by ω^2. Hence it is as follows: $(\rho_1\rho_6)(\rho_2\rho_5)(\rho_3\rho_4)$. As these two substitutions of order 2 have a product of order 3 they generate the dihedral group of order 6*, which is simply isomorphic with the symmetric group of order 6. This proves the following theorem: *the number* $\sqrt{-3} + p^{\frac{1}{3}}$ *generates a domain which belongs to the symmetric group of order* 6.

It remains only to construct a domain belonging to the group of order 7. This may readily be done by means of the complex roots of the equation $x^{29} - 1 = 0$. If θ is such a complex root it generates a domain which belongs to the cyclic group of order 28. Each of the seven distinct sets of sums of four roots corresponding to the subgroup of order 4 in this cyclic group of

* G. A. Miller, Bulletin of the American Mathematical Society, vol. 7 (1901), p. 424.

order 28 must therefore generate a domain which belongs to the group of order 7. As 12 belongs to exponent 4 mod 29, it results that one of these seven sets of numbers is $\theta + \theta^{12} + \theta^{28} + \theta^{17}$. That is, *if θ is a complex root of the equation $x^{29} - 1 = 0$ then will the number $\theta + \theta^{12} + \theta^{28} + \theta^{17}$ generate a domain which belongs to the group of order* 7.

2. Domains Belonging to Two Infinite Systems of Groups.

One of the most elementary infinite categories of groups is composed of all the possible groups which involve only operators of order 2, in addition to the identity. There is one and only one such group of order 2^m, m being an arbitrary positive rational integer, and all of these groups are abelian.* It is very easy to find a domain of rationality for each one of these groups. In fact, the number $\rho_1 = p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}} + \cdots + p_m^{\frac{1}{2}}$, where $p_1, p_2, \cdots, p_m$ are distinct rational prime numbers, generates a domain which belongs to this group of order 2^m. To prove that each of the 2^m numbers $\pm p_1^{\frac{1}{2}} \pm p_2^{\frac{1}{2}} \pm \cdots \pm p_m^{\frac{1}{2}}$ is a rational function of ρ_1 we observe that the first $2^m - 1$ powers of ρ_1 involve exactly $2^m - 1$ irrational numbers such that no two of them have a rational ratio, since the number of linear combinations of $p_1, p_2, \cdots, p_m$ is

$$m + \frac{m(m-1)}{2!} + \frac{m(m-1)(m-2)}{3!} + \cdots + m + 1 = (1+1)^m - 1 = 2^m - 1.$$

As ρ_1 is a root of an irreducible equation of degree 2^m, whose roots are $\pm p_1^{\frac{1}{2}} \pm p_2^{\frac{1}{2}} \pm \cdots \pm p_m^{\frac{1}{2}}$, it results that the given $2^m - 1$ powers of ρ_1 are linearly independent in the absolute domain of rationality. Hence the determinant of the system of $2^m - 1$ equations arising from these powers, the unknowns being the $2^m - 1$ combinations of $p_1^{\frac{1}{2}}, p_2^{\frac{1}{2}}, \cdots, p_m^{\frac{1}{2}}$, cannot vanish. That is, each of these unknowns is a rational function of ρ_1. In particular, each of the 2^m numbers $\pm p_1^{\frac{1}{2}} \pm p_2^{\frac{1}{2}} \pm \cdots \pm p_m^{\frac{1}{2}}$ is a rational function of ρ_1.

Suppose that each of these numbers is expressed rationally in terms of ρ_1. If we replace ρ_1, in each of these 2^m functions, by its value $p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}} + \cdots + p_m^{\frac{1}{2}}$, the coefficients of each of the $2^m - 1$ given unknowns, except those of $p_1^{\frac{1}{2}}, p_2^{\frac{1}{2}}, \cdots, p_m^{\frac{1}{2}}$, must vanish. If we replace ρ_1 by any other one of the roots $\pm p_1^{\frac{1}{2}} \pm p_2^{\frac{1}{2}} \pm \cdots \pm p_m^{\frac{1}{2}}$ in each of these 2^m functions, the same coefficients must evidently vanish, and the values of these functions can be obtained from those in which the value of ρ_1 was substituted by merely effecting the corresponding changes of signs in the coefficients of $p_1^{\frac{1}{2}}, p_2^{\frac{1}{2}}, \cdots, p_m^{\frac{1}{2}}$. As this is an operation of period two, it results that all the non-identical substitutions of the domain generated by ρ_1 must involve cycles of order 2.

* G. A. Miller, Quarterly Journal of Mathematics, vol. 28 (1896), p. 208.

As these substitutions are also regular, it follows that they all are of order 2, That is, *the domain of rationality generated by the number* $p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}} + \cdots + p_m^{\frac{1}{2}}$, *where* $p_1, p_2, \cdots, p_m$ *are distinct rational prime numbers, belongs to the abelian group of order* 2^m *and of type* $(1, 1, 1, \cdots)$.

Another very elementary infinite system of abelian groups is composed of all the groups which can be represented as groups of totitives; that is, all the groups formed by the $\varphi(m)$ positive rational integers which do not exceed m and are prime to m when these integers are combined by multiplication and the products are replaced by their least positive residues mod m, m being an arbitrary positive rational integer. This infinite category of groups may also be defined as composed of the groups of isomorphisms of all possible cyclic groups.

It is well known that the equation of degree $\varphi(m)$, whose roots are the $\varphi(m)$ primitive roots of the equation $x^m = 1$, is irreducible.* If θ represents one of these roots each of the other roots may be obtained by raising θ to the powers whose indices are the $\varphi(m)$ totitives of m. Hence it results that these roots are permuted according to this group of totitives if we express all of them in terms of θ and then replace θ in these expressions successively by each one of them. In other words, *any primitive mth root of unity generates a domain whose group is the group of the totitives of m.*

While it is thus easy to find a domain of rationality whose group is an arbitrary group of totitives, and to construct a domain for each one of a very interesting system of abelian groups, the most important matter related to this subject has not yet been mentioned. This may be stated as follows: *It is possible to find a group of totitives which has any arbitrary abelian group as one of its quotient groups.* Since a transitive abelian group permutes a set of systems of imprimitivity according to each one of its possible quotient groups, it results from the italicized theorem which has just been stated that *it is always possible to find sums of primitive roots of unity such that each of these sums generates a domain of rationality belonging to any arbitrary selected abelian group.* We proceed to establish these theorems.

To prove the former of these two theorems we need only combine the following two well known results: Every arithmetic progression in which the first term and the common difference are relatively prime involves an infinite number of prime numbers, and an abelian group has a quotient whose invariants are the same as the invariants of any given subgroup of this abelian group. From the former of these two theorems it results that there is an infinite number of different primes such that each of them diminished by unity is divisible by an arbitrary number n, and hence it is possible to find a number m such that the group of totitives involves an arbitrary number

* Cf. P. Bachmann, Die Lehre von der Kreisteilung, 1872, p. 321.

of independent cyclic subgroups of order n, n being an arbitrary positive rational integer. Hence it results from the latter of the given theorems that it is possible to find a group of totitives whose quotient group has an arbitrary set of positive rational integers as invariants.

If an abelian group is represented as a transitive substitution it must be regular and it must involve systems of imprimitivity corresponding to each one of its subgroups. In the group of totitives under consideration we can therefore add the roots which belong to the same system of imprimitivity and thus obtain a number which has the same number of conjugates as the order of the quotient group which corresponds to the subgroup under consideration. This general method was illustrated in the preceding section when domains of rationality belonging to the group of orders 5 and 7 were determined. From what precedes it results that this method can be used to construct a domain belonging to any given abelian group, as was stated above. In particular, we can also construct by this method a domain belonging to any one of the infinite systems of abelian groups of order 2^m and of type $(1, 1, 1, \cdots)$, which were considered above.

The proof of the theorem that every possible abelian group is contained in some group of isomorphisms of a cyclic group directs attention to the question whether every possible non-abelian group is contained in a group of isomorphisms of some non-cyclic abelian group. That this question can be answered in the affirmative is evident from the fact that in an abelian group of order p^m and of type $(1, 1, 1, \cdots)$ we can establish an isomorphism in which a particular set of independent generators corresponds to itself as a whole but permutes these generators according to an arbitrary substitution of degree m.

Hence it follows that the group of isomorphisms of this abelian group contains as a subgroup a group which is simply isomorphic with the symmetric group of degree m. As every possible group is simply isomorphic with some subgroup of a symmetric group, we have established, as a special case, the theorem that *every possible group is isomorphic with a subgroup in the group of isomorphisms of an abelian group.*

ON LEBESGUE'S CONSTANTS IN THE THEORY OF FOURIER'S SERIES.*

By T. H. Gronwall.

The definition of the nth Lebesgue constant ρ_n is

$$\rho_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left|\frac{\sin(2n+1)t}{\sin t}\right|dt,$$

and by certain transformations of this integral, I have shown in a previous paper† that

$$\rho_{n+1} > \rho_n, \qquad (n = 1, 2, 3, \cdots). \tag{1}$$

In the latter part of the paper referred to, I used Fejér's formula

$$\rho_n = \frac{2}{\pi}\sum_{\mu=1}^{n}\frac{1}{\mu}\tan\frac{\mu\pi}{2n+1} + \frac{1}{2n+1} \tag{2}$$

to obtain a general asymptotic expression for ρ_n; and from the point of view of unity in method, it is desirable also to deduce (1) directly from (2). This is the purpose of the present note.

From (2) we obtain at once

$$\begin{aligned}\rho_{n+1} - \rho_n &= \frac{2}{\pi}\cdot\frac{1}{n+1}\tan\frac{n+1}{2n+3}\pi + \frac{2}{\pi}\sum_{\mu=1}^{n}\frac{1}{\mu}\left(\tan\frac{\mu\pi}{2n+3} - \tan\frac{\mu\pi}{2n+1}\right)\\ &\qquad + \frac{1}{2n+3} - \frac{1}{2n+1}\\ &= \frac{2}{\pi}\cdot\frac{1}{n+1}\cot\frac{\pi}{2(2n+3)} - \frac{2}{\pi}\sum_{\mu=1}^{n}\frac{1}{\mu}\cdot\frac{\sin\dfrac{2\mu\pi}{(2n+1)(2n+3)}}{\cos\dfrac{\mu\pi}{2n+1}\cos\dfrac{\mu\pi}{2n+3}}\\ &\qquad - \frac{2}{(2n+1)(2n+3)}\\ &> \frac{2}{\pi}\cdot\frac{1}{n+1}\cot\frac{\pi}{2(2n+3)}\\ &\qquad - \frac{4}{(2n+1)(2n+3)}\sum_{\mu=1}^{n}\frac{1}{\cos\dfrac{\mu\pi}{2n+1}\cos\dfrac{\mu\pi}{2n+3}}\\ &\qquad - \frac{2}{(2n+1)(2n+3)},\end{aligned}$$

* Presented to the American Mathematical Society, February 22, 1913.

† "Uber die Lebesgueschen Konstanten bei den Fourierschen Reihen," Math. Annalen, vol. 72 (1912), pp. 244–261.

or introducing the summation index $\nu = n - \mu$,

$$\begin{aligned}\rho_{n+1} - \rho_n > \frac{2}{\pi}\frac{1}{n+1}\cot\frac{\pi}{2(2n+3)} \\ - \frac{4}{(2n+1)(2n+3)}\sum_{\nu=0}^{n-1}\frac{1}{\sin\frac{2\nu+1}{2n+1}\frac{\pi}{2}\sin\frac{2\nu+3}{2n+3}\frac{\pi}{2}} \\ - \frac{2}{(2n+1)(2n+3)}.\end{aligned} \tag{3}$$

From the formula

$$\frac{1}{\sin x} = \cot\frac{x}{2} - \cot x$$

and the development of the cotangent in partial fractions it is seen that

$$\frac{1}{x}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi^2 - x^2} - \frac{1}{n^2\pi^2 - x^2/4}\right),$$

and since each term in the series increases monotonely with x for $0 < x < \pi/2$, the same is true for the left hand expression, so that

$$\frac{1}{x}\left(\frac{1}{\sin x} - \frac{1}{x}\right) < \frac{2}{\pi}\left(1 - \frac{2}{\pi}\right) \qquad \left(0 < x < \frac{\pi}{2}\right),$$

or denoting the right hand constant by a,

$$\frac{1}{\sin x} < \frac{1}{x} + ax \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{4}$$

In the same way, it is seen that

$$\frac{1}{x}\left(\frac{1}{x} - \cot x\right) = \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2 - x^2}$$

increases monotonely with x for $0 < x < \pi/2$, so that

$$\cot x > \frac{1}{x} - \frac{4}{\pi^2}x \qquad \left(0 < x < \frac{\pi}{2}\right).$$

We have, therefore,

$$\begin{aligned}\frac{2}{\pi}\cdot\frac{1}{n+1}\cot\frac{\pi}{2(2n+3)} > \frac{4}{\pi^2}\cdot\frac{2n+3}{n+1} - \frac{4}{\pi^2}\cdot\frac{1}{(n+1)(2n+3)} \\ = \frac{4}{\pi^2}\cdot\frac{4n+8}{2n+3} = \frac{8}{\pi^2} + \frac{8}{\pi^2}\cdot\frac{1}{2n+1} - \frac{16}{\pi^2}\cdot\frac{1}{(2n+1)(2n+3)}.\end{aligned} \tag{5}$$

Now (4) gives

$$\frac{4}{(2n+1)(2n+3)}\cdot\frac{1}{\sin\dfrac{2\nu+1}{2n+1}\dfrac{\pi}{2}\sin\dfrac{2\nu+3}{2n+3}\dfrac{\pi}{2}}<\frac{16}{\pi^2}\frac{1}{(2\nu+1)(2\nu+3)}$$

$$+4a\left(\frac{1}{(2n+1)^2}\cdot\frac{2\nu+1}{2\nu+3}+\frac{1}{(2n+3)^2}\cdot\frac{2\nu+3}{2\nu+1}\right)+a^2\pi^2\frac{(2\nu+1)(2\nu+3)}{(2n+1)^2(2n+3)^2}.$$

We have

$$\sum_{\nu=0}^{n-1}\frac{16}{\pi^2}\cdot\frac{1}{(2\nu+1)(2\nu+3)}=\frac{8}{\pi^2}\sum_{\nu=0}^{n-1}\left(\frac{1}{2\nu+1}-\frac{1}{2\nu+3}\right)=\frac{8}{\pi^2}\left(1-\frac{1}{2n+1}\right)$$

and

$$\frac{1}{(2n+1)^2}\cdot\frac{2\nu+1}{2\nu+3}+\frac{1}{(2n+3)^2}\cdot\frac{2\nu+3}{2\nu+1}=\frac{1}{(2n+1)^2}-\frac{2}{(2n+1)^2}\cdot\frac{1}{2\nu+3}$$

$$+\frac{1}{(2n+3)^2}+\frac{2}{(2n+3)^2}\cdot\frac{1}{2\nu+1}<\frac{1}{(2n+1)^2}+\frac{1}{(2n+3)^2}$$

$$+\frac{2}{(2n+3)^2}\left(\frac{1}{2\nu+1}-\frac{1}{2\nu+3}\right),$$

whence

$$\sum_{\nu=0}^{n-1}\left(\frac{1}{(2n+1)^2}\cdot\frac{2\nu+1}{2\nu+3}+\frac{1}{(2n+3)^2}\cdot\frac{2\nu+3}{2\nu+1}\right)<\frac{n}{(2n+1)^2}+\frac{n}{(2n+3)^2}$$

$$+\frac{2}{(2n+3)^2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2n+1}-\frac{1}{(2n+1)(2n+3)}$$

$$-\frac{8n+6}{(2n+1)^2(2n+3)^2}<\frac{1}{2n+1}-\frac{1}{(2n+1)(2n+3)}.$$

Furthermore we have

$$\frac{1}{(2n+1)^2(2n+3)^2}\sum_{\nu=0}^{n-1}(2\nu+1)(2\nu+3)$$

$$=\frac{1}{(2n+1)^2(2n+3)^2}\sum_{\nu=0}^{n-1}\frac{(2\nu+1)(2\nu+3)(2\nu+5)-(2\nu-1)(2\nu+1)(2\nu+3)}{6}$$

$$=\frac{1}{(2n+1)^2(2n+3)^2}\cdot\frac{(2n-1)(2n+1)(2n+3)+3}{6}$$

$$=\frac{1}{6}\left(\frac{1}{2n+1}-\frac{4}{(2n+1)(2n+3)}+\frac{3}{(2n+1)^2(2n+3)^2}\right)$$

$$\leqq\frac{1}{6}\cdot\frac{1}{2n+1}-\frac{1}{2}\cdot\frac{1}{(2n+1)(2n+3)},$$

so that we finally obtain

$$
(6)\quad \begin{aligned}
&\frac{4}{(2n+1)(2n+3)} \sum_{\nu=0}^{n-1} \frac{1}{\sin \frac{2\nu+1}{2n+1}\frac{\pi}{2} \sin \frac{2\nu+3}{2n+3}\frac{\pi}{2}} < \frac{8}{\pi^2}\left(1 - \frac{1}{2n+1}\right) \\
&\quad + \frac{4a}{2n+1} - \frac{4a}{(2n+1)(2n+3)} + \frac{a^2\pi^2}{6} \cdot \frac{1}{2n+1} - \frac{a^2\pi^2}{2} \frac{1}{(2n+1)(2n+3)} .
\end{aligned}
$$

From (3), (5) and (6) it is now seen that

$$
\rho_{n+1} - \rho_n > \frac{1}{2n+1}\left[\left(\frac{16}{\pi^2} - 4a - \frac{a^2\pi^2}{6}\right) - \left(\frac{16}{\pi^2} + 2 - 4a - \frac{a^2\pi^2}{2}\right)\frac{1}{2n+3}\right],
$$

or introducing

$$
a = \frac{2}{\pi}\left(1 - \frac{2}{\pi}\right)
$$

and calculating the numerical values of the constants

$$
(7)\qquad \rho_{n+1} - \rho_n > \frac{1}{2n+1}\left(0.607 - \frac{2.432}{2n+3}\right).
$$

This expression being evidently positive for $n \geqq 1$, our formula (1) is proved.

PRINCETON UNIVERSITY.

THE LINEAR DIFFERENCE EQUATION OF THE FIRST ORDER.*

By K. P. Williams.

In this paper we consider the first order linear difference equation

$$(A) \qquad f(x+1) = E(x)f(x),$$

where $E(x)$ is an entire function. It will be seen that the theory for such an equation gives at once the theory for the equation where the coefficient of $f(x)$ is a meromorphic function.

The equation (A) was first completely solved by Hurwitz,† who showed the existence of a solution when $E(x)$ is any entire function. He did not, however, consider this equation primarily, but rather the equation

$$(B) \qquad f(x+1) - f(x) = \varphi(x),$$

for which he proved that there was an entire solution in case $\varphi(x)$ is entire,‡ and a meromorphic solution in case $\varphi(x)$ is meromorphic. The logarithmic derivative of the first equation reduces to one of the form (B), where the known function is meromorphic. Thus the solution Hurwitz gave of (B) included that of our equation.

Somewhat later Barnes§ gave a direct treatment of equation (A), by expressing the function $E(x)$ in its Weierstrass factor form. To include all cases he found it necessary, however, to know that equation (B) has a solution when $\varphi(x)$ is an entire function. But Hurwitz proved in an elegant way that to pass from the case where $\varphi(x)$ in (B) is entire to the case where it is meromorphic, is but a simple step, and this, as remarked above, is to know completely the theory for (A). Thus we gain nothing if we appeal at all to (B) in giving a direct solution for the first equation. The solution that Barnes obtained is, furthermore, to a large extent undetermined; for it contains certain functions of the zeros of $E(x)$, whose explicit forms, he states, he was able to determine only in certain simple cases.

The present discussion also depends fundamentally on the factor form of $E(x)$; but the method is essentially different from that used by Barnes. The solution which is obtained is simple, and is completely determined.

* Presented to the American Mathematical Society, December 27, 1913.

† Acta Mathematica (1897), T. XX, pp. 285–312.

‡ An interesting discussion of (B) for the case where $\varphi(x)$ is entire is given by Carmichael, Amer. Journal of Math., vol. XXXV, pp. 165–182.

§ Proceedings of London Math. Soc. (1904), 2d series, vol. 2, pp. 438–469. Barnes considers the equation $f(x+\omega) = \varphi(x)f(x)$, where ω is any real or complex constant. This is no more general, and adds only to the complexity of the formulæ.

We restrict ourselves entirely to those cases which do not need to be made to depend on (B); this includes all cases where $E(x)$ is an entire function of finite genus, and also a large class of cases where this limitation is not fulfilled. It is also thought that the present method has the advantage of showing clearly why a simple choice of certain convergence factors can be made.

Let $a_1, a_2, \cdots$, be the zeros of $E(x)$; then we know by Weierstrass's factor theorem that we can always write

$$E(x) = e^{g(x)} \prod_{i=1}^{\infty} \left(1 - \frac{x}{a_i}\right) e^{\sum_{n=1}^{i-1} \frac{1}{n} \left(\frac{x}{a_i}\right)^n},$$

where $g(x)$ is itself some entire function. If we can find an integer k such that the series

$$\sum_{i=1}^{\infty} \left|\frac{1}{a_i}\right|^{k+1}$$

is convergent, we say the function $E(x)$ is of finite order, and we have the simpler form

$$E(x) = e^{g(x)} \prod_{i=1}^{\infty} \left(1 - \frac{x}{a_i}\right) e^{\sum_{n=1}^{k} \frac{1}{n} \left(\frac{x}{a_i}\right)^n}.$$

If further $g(x)$ reduces to a polynomial of degree q, we say that the function $E(x)$ is of finite genus r, where r is the greatest of k and q, provided k is the smallest positive integer for which the above series is convergent. We shall limit ourselves to the cases where $g(x)$ is a polynomial, the order being either finite or infinite.

The resolution of $E(x)$ into factors suggests at once a method for solving our equation; for a fundamental property of such an equation is that the solution of

$$f(x + 1) = u_1(x)u_2(x)f(x),$$

where $u_1(x)$ and $u_2(x)$ are known, is found by forming the product of the solutions of the two equations

$$f(x + 1) = u_1(x)f(x), \quad \text{and} \quad f(x + 1) = u_2(x)f(x).$$

In order to be able to write the solutions corresponding to the exponential factors in $E(x)$ we note first that a solution of the simple equation

$$f(x + 1) = e^{ax^n} f(x),$$

where a is any constant, and n a positive integer, is given immediately by

$$f(x) = e^{\frac{a\phi_{n+1}(x)}{n+1}},$$

where $\varphi_{n+1}(x)$ is the nth Bernoulli polynomial. The fundamental prin-

ciple mentioned above then shows at once how to solve the equation $f(x+1) = e^{v(x)}f(x)$, where $v(x)$ is any polynomial. In the cases we are considering we can thus solve easily the equation

$$f(x+1) = e^{g(x)}f(x),$$

which arises from the extraneous exponential factor in $E(x)$; we indicate the solution by $e^{\bar{g}(x)}$, $\bar{g}(x)$ being a polynomial which can be explicitly expressed in terms of the Bernoulli polynomials and the constants in $g(x)$.

If now we assume that equation (B) has a solution whenever $\varphi(x)$ is entire, we see that the equation which we have last written has a solution even in case $g(x)$ ceases to be a polynomial, but is any undetermined entire function. It was precisely at this point that Barnes introduced (B) into his consideration of (A).

We notice finally that a solution of the equation $f(x+1) = (1-x/a_i)f(x)$, is given by $f(x) = \Gamma(x-a_i)(-a_i)^{-x}$, $\Gamma(x)$ being the gamma function. We thus have solutions corresponding to all the types of factors in $E(x)$; and from these we build up a solution of (A).

Let us first assume that the function $E(x)$ is of finite order. The equation then takes the form

$$f(x+1) = f(x)e^{g(x)}\prod_{i=1}^{\infty}\left(1-\frac{x}{a_i}\right)e^{\sum_{n=1}^{k}\frac{1}{n}\left(\frac{x}{a_i}\right)^n},$$

the series $\sum_{i=1}^{\infty}\left|\frac{1}{a_i}\right|^{k+1}$ being convergent.

A formal solution is at once seen to be

$$f(x) = e^{\bar{g}(x)}\prod_{i=1}^{\infty}\frac{\Gamma(x-a_i)e^{\theta_i(x)}}{(-a_i)^x p_i},$$

where

$$\theta_i(x) = \frac{\varphi_2(x)}{1\cdot 2a_i} + \frac{\varphi_3(x)}{2\cdot 3a_i^2} + \cdots + \frac{\varphi_{k+1}(x)}{k(k+1)a_i^k},$$

while $p_1, p_2, \cdots$ are any constants (or periodic functions of period 1).

If now we can choose the quantities p_i so that the above expression is a uniformly convergent product we shall have an analytic solution of our equation. A method for doing this will be found by means of the asymptotic form of the gamma function.

Before we proceed, however, we shall introduce a certain convention which will save unnecessary repetition. We have frequent occasion to use functions of x and a_i, say $M(x, a_i)$, which have the property of being less in absolute value than some properly chosen constant, provided x is confined to any limited region, and $|a_i|$ is large enough. Since $|a_i|$ increases indefinitely with i, the last condition will be fulfilled if i is sufficiently

great, say if $i > I$. The value of I depends in general on the region in which x can vary. Under these conditions we shall merely say that $M(x, a_i)$ is bounded, *always* understanding by that term that x and i satisfy the requisite conditions; that is, x is in some limited region, while i is larger than some positive integer, depending perhaps on the region in which x can vary. Similarly, when we say a function of a_i alone is bounded, we shall *always* mean that it is less in absolute value than some constant if $|a_i|$ is sufficiently great, that is, if we choose i large enough. Such functions as those described will occur in our work upon breaking off convergent series, or series which asymptotically represent a function.

In order to be able to expand $\Gamma(x - a_i)$, we shall assume for the present that all the points $a_1, a_2, \cdots$ lie to the left of a certain line parallel to the imaginary axis, and are arranged in order of increasing absolute values. We can then use Stirling's formula for the gamma function, and we thus obtain

$$\Gamma(x - a_i) = \sqrt{2\pi}(x - a_i)^{x-a_i-\frac{1}{2}}e^{-x+a_i+\psi(x-a_i)},$$

where

$$\psi(x - a_i) = \frac{B_1}{1 \cdot 2(x - a_i)} - \frac{B_2}{3 \cdot 4(x - a_i)^3} + \cdots + \frac{(-1)^{t+1}B_t}{(2t - 1)2t(x - a_i)^{2t-1}} + \frac{R_t(x, a_i)}{(x - a_i)^{2t+1}},$$

the B's being the Bernoulli numbers, t any positive integer, and $R_t(x, a_i)$ a bounded function.

Therefore if i is sufficiently great we have for the general term of our formal product

$$\frac{\Gamma(x - a_i)e^{\theta_i(x)}}{(-a_i)^x p_i} = \frac{\sqrt{2\pi}(-a_i)^{-a_i-\frac{1}{2}}}{p_i}e^{\omega_i(x)},$$

where, for short, we have put

$$\omega_i(x) = a_i - x + \left(x - a_i - \frac{1}{2}\right)\log\left(1 - \frac{x}{a_i}\right) + \theta_i(x) + \psi(x - a_i).$$

We next expand $\omega_i(x)$ in terms of powers of $1/a_i$. In the first place we have

$$- x + \left(x - a_i - \frac{1}{2}\right)\log\left(1 - \frac{x}{a_i}\right)$$

$$= -\frac{x^2 - x}{1 \cdot 2a_i} - \frac{\left(x^3 - \frac{3}{2}x^2\right)}{2 \cdot 3a_i^2} - \cdots - \frac{\left(x^{k+1} - \frac{k+1}{2}x^k\right)}{k(k+1)a_i^k} + \frac{\eta_k(x, a_i)}{a_i^{k+1}},$$

where $\eta(x, a_i)$ is bounded.

The function $\theta_i(x)$ is already a polynomial in $1/a_i$, so that we need merely transform $\psi(x - a_i)$.

To find the desired expression for $\psi(x - a_i)$ let us formally expand the divergent infinite series

$$\frac{B_1}{1 \cdot 2(x - a_i)} - \frac{B_2}{3 \cdot 4(x - a_i)^3} + \frac{B_3}{5 \cdot 6(x - a_i)^5} - \cdots,$$

as a series in $1/a_i$. We obtain for the typical term the expression

$$\frac{1}{s(s+1)a_i^s}\left\{ - \binom{s+1}{2} B_1 x^{s-1} + \binom{s+1}{4} B_2 x^{s-3} + \cdots \right\},$$

where the quantity in parentheses ends with $(-1)^{(s+1)/2} B_{(s+1)/2}$, if s is odd, and in the term in x if s is even. We can simplify this result by making use of the Bernoulli polynomials.

We know that

$$\varphi_n(x) = x^n - \frac{n}{2} x^{n-1} + \binom{n}{2} B_1 x^{n-2} - \binom{n}{4} B_2 x^{n-4} + \cdots,$$

the expression terminating with the term in x or x^2. The general term of the above expansion can thus be written

$$\frac{1}{s(s+1)a_i^s}\left\{ x^{s+1} - \frac{s+1}{2} x^s - \varphi_{s+1}(x) + (-1)^{\frac{s+1}{2}} B_{\frac{s+1}{2}} \right\},$$

if we agree to put $B_{(s+1)/2}$ equal to zero when s is even.

Let us next choose the arbitrary integer t which occurs in $\psi(x - a_i)$ so that $2t + 1 \geqq k + 1$. It follows then, by making use of the expression last obtained, that we can write

$$\begin{aligned} \psi(x - a_i) = {} & \frac{1}{1 \cdot 2a_i}\{x^2 - x - \varphi_2(x) - B_1\} + \frac{1}{2 \cdot 3a_i^2}\left\{ x^3 - \frac{3}{2} x^2 - \varphi_3(x) \right\} \\ & + \cdots + \frac{1}{k(k+1)a_i^k}\left\{ x^{k+1} - \frac{k+1}{2} x^k - \varphi_{k+1}(x) + (-1)^{\frac{k+1}{2}} B_{\frac{k+1}{2}} \right\} \\ & + \frac{R(x, a_i)}{a_i^{k+1}}, \end{aligned}$$

where $R(x, a_i)$ is bounded.

We now have all the quantities which occur in $\omega_i(x)$ written in terms of $1/a_i$. Adding them we find the simple result

$$\omega_i(x) = a_i - \frac{B_1}{1 \cdot 2a_i} + \frac{B_2}{3 \cdot 4a_i^3} - \frac{B_3}{5 \cdot 6a_i^5} + \cdots + \frac{(-1)^{\frac{k+1}{2}} B_{\frac{k+1}{2}}}{k(k+1)a_i^k} + \frac{M(x, a_i)}{a_i^{k+1}},$$

where $M(x, a_i) = \eta(x, a_i) + R(x, a_i)$ is bounded.

The essential property of the expression which we have derived for $\omega_i(x)$ is that the terms involving $1/a_i$, $1/a_i^3$, $\cdots$, $1/a_i^k$, do not contain the variable x,

and are, moreover, the first terms in the series which occurs in the asymptotic form of $\Gamma(-a_i)$. This fact enables us to determine in a simple manner suitable choices for $p_1, p_2, \cdots$.

Let us put

$$p_i = \Gamma(-a_i) = \sqrt{2\pi}(-a_i)^{-a_i-\frac{1}{2}}e^{\xi(a_i)},$$

where

$$\xi(a_i) = a_i - \frac{B_1}{1 \cdot 2a_i} + \frac{B_2}{3 \cdot 4a_i^3} - \cdots + \frac{(-1)^{\frac{k+1}{2}}B_{\frac{k+1}{2}}}{k(k+1)a_i^k} + \frac{\epsilon(a_i)}{a_i^{k+1}},$$

$\epsilon(a_i)$ being bounded.

With this selection the typical term of our infinite product is, for large enough i,

$$\frac{\Gamma(x-a_i)e^{\theta_i(x)}}{(-a_i)^x\Gamma(-a_i)} = e^{\frac{M'(x,\, a_i)}{a_i^{k+1}}},$$

where $M'(x, a_i)$ is bounded.

What we have proved is this: *As soon as the region in which x can vary is known we can find a positive integer I, and a positive constant K, such that, if $p_i = \Gamma(-a_i)$, the general term of our product is given by the last formula, in which $|M'(x, a_i)| < K$, provided only that $i > I$.*

But we know that the series

$$\sum_{i=1}^{\infty}\left|\frac{1}{a_i^{k+1}}\right|$$

is convergent, and we thus see at once that the selection we have made for p_i makes our product converge uniformly in any region where its factors are all analytic.

We have then as a solution of the given equation the function

$$f(x) = e^{\bar{g}(x)}\prod_{i=1}^{\infty}\frac{\Gamma(x-a_i)e^{\sum_{n=2}^{k+1}\frac{\phi_n(x)}{(n-1)na_i^{n-1}}}}{(-a_i)^x\Gamma(-a_i)}.$$

This solution is seen to have no zeros in the finite plane, while the nature and situation of its singularities follow at once from the theory of the gamma function.

Let us take next the case where the zeros of $E(x)$ are such that they lie to the right of a line parallel to the axis of imaginaries. We obtain a solution by merely everywhere replacing the gamma function in the above result by the second gamma function; the second gamma function, $\bar{\Gamma}(u)$, being the solution of

$$f(x+1) = xf(x),$$

that is asymptotic to Stirling's series in the left half plane; it is connected with $\Gamma(u)$ by the relation $\bar{\Gamma}(u) = (1 - e^{2\pi u\sqrt{-1}})\Gamma(u)$.

Consider now the case where $E(x)$ is any entire function of finite genus. To apply the results already obtained we divide the zeros into two groups; those on or to the left of the imaginary axis forming one group, and those on the right the other. Let us call these two groups $a_1, a_2, \cdots$, and $b_1, b_2, \cdots$, respectively. The solution is then found in a perfectly obvious way as the product of the solutions of two equations of the types we have considered, and is

$$f(x) = e^{\bar{g}(x)} \prod_{i=1}^{\infty} \frac{\Gamma(x - a_i) e^{\sum_{n=2}^{k+1} \frac{\phi_n(x)}{(n-1)n a_i^{n-1}}}}{(-a_i)^x \Gamma(-a_i)} \prod_{i=1}^{\infty} \frac{\bar{\Gamma}(x - b_i) e^{\sum_{n=2}^{k+1} \frac{\phi_n(x)}{(n-1)n b_i^{n-1}}}}{(-b_i)^x \bar{\Gamma}(-b_i)}.$$

Since the second gamma function, $\bar{\Gamma}(u)$, is zero for $u = 1, 2, \cdots$, our solution is a meromorphic function, with its zeros at the points

$$x = b_i + n \qquad (i, n = 1, 2, \cdots),$$

and its poles at the points

$$x = a_i - n \qquad (i = 1, 2, \cdots, n = 0, 1, 2, \cdots).$$

So far we have assumed explicitly that $E(x)$ is of finite order. Let us now suppose that this is no longer the case, but that the function $g(x)$ is still a polynomial. By making a modification of our reasoning exactly similar to that for finding the Weierstrass primary factors for the corresponding case we derive a solution, which is obtained from the one above, for the case of finite order, by making the summation which appears as the exponent of e in the ith factor include terms from $n = 2$ to $n = i$.

We shall finally consider briefly the case where the coefficient of $f(x)$ in (A) is a meromorphic function. The results which we have already obtained will enable us to show the existence of a solution provided a certain restriction is fulfilled.

Suppose first that the coefficient is the reciprocal of an entire function. The substitution $f(x) = 1/f(x)$ reduces the equation at once to the equation we have considered. This enables us to treat the general case; for we know that any meromorphic function is the quotient of two entire functions. Let us denote the extraneous exponential factor in these two functions by $e^{g_1(x)}$ and $e^{g_2(x)}$, respectively. Then if $g_1(x) - g_2(x)$ reduces to a polynomial we see that the solution can be written as the quotient of the solutions of two equations, each of which we know can be solved. The solution will be a meromorphic function, with its zeros and poles at known points.

The class of entire and meromorphic functions for which we have obtained a simple and completely determined solution of (A) is seen to be a very wide one.

BLOOMINGTON, INDIANA,
June, 1913.

GEOMETRIC PROPERTIES OF THE JACOBIANS OF A CERTAIN SYSTEM OF FUNCTIONS.*

By Arnold Emch.

1. In the proofs of the existence theorem for implicit functions of several variables the assumption is made that the corresponding Jacobian does not vanish for any point within the interval for which the functions are defined.† Also in the general theory of analysis situs as developed so far,‡ the cases in which the Jacobians vanish simultaneously with the corresponding functions are excluded.

It may be expected that the vanishing of a Jacobian and its functions for certain values of the variables signifies a particular property of these functions which deserves to be investigated.§

It is the purpose of this paper to show the importance of such cases by studying the geometric properties of a certain system of functions and their Jacobians.

2. Let

$$x = \phi(t), \qquad y = \psi(t) \tag{1}$$

be two real, distinct, uniform, continuous and singly periodic functions of the real parameter t and with the same period w. As is well known, in a Cartesian plane (1) represents a closed *Jordan curve*.‖

We assume furthermore that $\phi'(t)$ and $\psi'(t)$ are also continuous within the period-interval and do not vanish simultaneously for any value of t. In other words, the curve as represented by (1) is analytic throughout and its only singular points, if there are any, are multiple points.

3. If z_1, z_2, z_3, z_4 designate four independent variables, consider now the three functions

$$\tag{2} \begin{cases} F_1(z_1, z_2, z_3, z_4) = \phi(z_1) - \phi(z_2) + \phi(z_3) - \phi(z_4), \\ F_2(z_1, z_2, z_3, z_4) = \psi(z_1) - \psi(z_2) + \psi(z_3) - \psi(z_4), \\ F_3(z_1, z_2, z_3, z_4) = [\phi(z_1) - \phi(z_3)][\phi(z_2) - \phi(z_4)] \\ \qquad\qquad + [\psi(z_1) - \psi(z_3)][\psi(z_2) - \psi(z_4)]. \end{cases}$$

* Read before the American Mathematical Society in Chicago, Dec. 26, 1913.

† Genocchi-Peano, Differentialrechnung (1899), pp. 147–152; Bliss, "A new proof of the existence theorem of implicit functions," Bulletin of the Am. Math. Soc., vol. XVIII, pp. 175–179 (1912).

‡ Poincaré: "Analysis situs," Journal de l'École Polytéchnique, 2nd ser., vol. 1, pp. 1–121.

§ Clements, "Implicit functions defined by equations with vanishing Jacobian," Bulletin of the Am. Math. Soc., vol. XVIII, pp. 451–456 (1912).

‖ "Osgood, Lehrbuch der Funktionentheorie," vol. 1, 2nd ed., pp. 146–150 (1913).

They are evidently uniform, continuous, analytic and co-periodic for all values of the variables. To any four values $z_1 < z_2 < z_3 < z_4$ within a period-interval correspond on the curve (1) the vertices of an inscribed quadrangle $A_1A_2A_3A_4$ which follow each other in the same order as the entire curve is described in the same sense. If for any set of values of the z's $F_1 = 0$, $F_2 = 0$, $F_3 = 0$, then the quadrangle will be a rhomb. As will be proved elsewhere, there are always an infinite number of rhombs inscribable in the given curve. For any given direction and the corresponding orthogonal direction there is at least one rhomb inscribable and with its diagonals parallel to the pair of orthogonal directions. There is always a continuous curve between the points of tangency of the two extreme tangents to the curve parallel to the given direction and which is the locus of mid-points of a continuous system of chords parallel to the same direction. I shall call such a curve a *median* M_τ of the given closed curve C (1), with respect to the direction τ. *If C_1C_2 is one of the chords of the system cutting M_τ in M, then the tangents to C at C_1 and C_2 and to M_τ at M are concurrent.*

With every pair τ, σ of orthogonal directions are associated two medians M_τ and M_σ which always intersect in at least one point. The points of intersection of M_τ and M_σ are evidently the centers of inscribed rhombs with diagonals parallel to τ and σ. *When M_τ and M_σ touch each other at some point R, then the tangents to C at A_1 and A_3 intersect in a point T_1 and those at A_2, A_4 in a point T_2 so that T_1, R, T_2 are collinear.*

4. In the system of simultaneous equations

$$F_1 = 0, \qquad F_2 = 0, \qquad F_3 = 0, \tag{3}$$

consider any one of the four variables, for instance z_4 as independent. According to the theorem on implicit functions,* in the neighborhood of any set z_1, z_2, z_3, z_4 satisfying (3) and for which the Jacobian

$$(4)\quad \frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)}$$

$$= \begin{vmatrix} \phi'(z_1) & -\phi'(z_2) & \phi'(z_3) \\ \psi'(z_1) & -\psi'(z_2) & \psi'(z_3) \\ \left\{\begin{array}{l} \phi'(z_1)[\phi(z_2)-\phi(z_4)] \\ +\psi'(z_1)[\psi(z_2)-\psi(z_4)] \end{array}\right\}, & \left\{\begin{array}{l} \phi'(z_2)[\phi(z_1)-\phi(z_3)] \\ +\psi'(z_2)[\psi(z_1)-\psi(z_3)] \end{array}\right\}, & \left\{\begin{array}{l} -\phi'(z_3)[\phi(z_2)-\phi(z_4)] \\ -\psi'(z_3)[\psi(z_2-\psi(z_4)] \end{array}\right\} \end{vmatrix}$$

does not vanish, it is possible to represent z_1, z_2, z_3 as uniform, continuous and analytic functions of z_4.

* Bliss, loc. cit.

Equations (3) and (4) vanish simultaneously only for a finite number of sets z_1, z_2, z_3, z_4; hence for any domain of four space z_1, z_2, z_3, z_4 within which

$$F_1 = 0, \qquad F_2 = 0, \qquad F_3 = 0,$$

and

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)} \neq 0,$$

the z_i's ($i = 1, 2, 3, 4$) form a continuous set. Geometrically to such a domain corresponds a continuous set of inscribed rhombs.

5. I shall now investigate the case in which the Jacobian (4) vanishes simultaneously with (3). Without loss of generality it may be assumed that the axes of the rhomb in this case coincide with the coordinate axes, so that $A_1(z_1)$, $A_3(z_3)$ are on the x- and $A_2(z_2)$, $A_4(z_4)$ on the y-axis. Designating the coordinates of A_1, A_2, A_3, A_4 respectively by $(a, 0)$; $(0, b)$; $(-a, 0)$; $(0, -b)$, the condition that (4) vanishes becomes

$$\begin{vmatrix} \phi'(z_1) & -\phi'(z_2) & \phi'(z_3) \\ \psi'(z_1) & -\psi'(z_2) & \psi'(z_3) \\ 2b\psi'(z_1) & 2a\phi'(z_2) & -2b\psi'(z_3) \end{vmatrix} = 0. \tag{5}$$

Assuming $\phi'(z_i) \neq 0$ ($i = 1, 2, 3$) which, by choosing the x- and y-axis properly, does not imply a special case; dividing the first, second and third column by $\phi'(z_1)$, $\phi'(z_2)$, $\phi'(z_3)$ respectively, and designating the slopes of the tangents to C at A_1, A_2, A_3 by m_1, m_2, m_3, (5) reduces to

$$\phi'(z_1)\phi'(z_2)\phi'(z_3)\begin{vmatrix} 1 & -1 & 1 \\ m_1 & -m_2 & m_3 \\ bm_1 & a & -bm_3 \end{vmatrix} = 0, \tag{6}$$

or explicitly, if $\phi'(z_1) \neq 0$, $\phi'(z_2) \neq 0$, $\phi'(z_3) \neq 0$, to

$$a(m_1 - m_3) + b(m_1m_2 - 2m_1m_3 + m_2m_3) = 0. \tag{7}$$

6. To find a geometric interpretation for this equation, consider the corresponding rhomb A_1, A_2, A_3, A_4 and apply first the dilatation

$$\begin{aligned} x' &= (1 + e_1)x \\ y' &= (1 + e_2)(y + b) - b, \end{aligned} \tag{8}$$

and second the rotation

$$\begin{aligned} x'' &= x' \cos\theta - (y' + b)\sin\theta \\ y'' &= x' \sin\theta + (y' + b)\cos\theta \end{aligned} \tag{9}$$

with A_4 as a center, Fig. 1. The lines $x = 0$ and $y = -b$ are invariant in the dilatation. After this dilatation and rotation the coordinates of the rhomb in the new position are for

$$A_1''\begin{cases}(1+e_1)a\cos\theta-(1+e_2)b\sin\theta,\\(1+e_1)a\sin\theta+(1+e_2)b\cos\theta-b,\end{cases}$$

$$A_2''\begin{cases}-(1+e_2)2b\sin\theta,\\(1+e_2)2b\cos\theta-b,\end{cases}$$

$$A_3''\begin{cases}-(1+e_1)a\cos\theta-(1+e_2)b\sin\theta,\\-(1+e_1)a\sin\theta+(1+e_2)b\cos\theta-b,\end{cases}$$

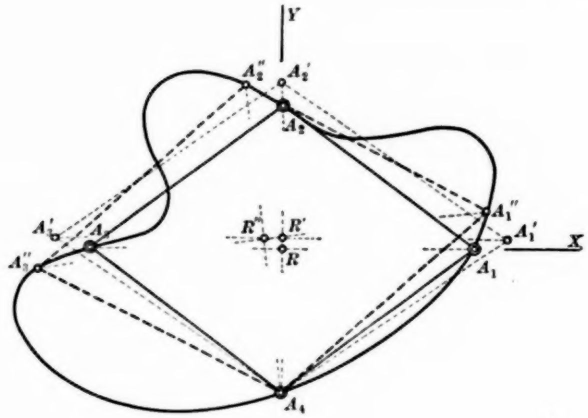


FIG. 1.

referred to the original xy-plane. The slopes μ_1, μ_2, μ_3 of the lines A_1A_1'', A_2A_2'', A_3A_3'' are

$$\mu_1=\frac{(1+e_1)a\sin\theta+(1+e_2)b\cos\theta-b}{(1+e_1)a\cos\theta-(1+e_2)b\sin\theta-a},$$

$$u_2=\frac{(1+e_2)b\cos\theta-b}{(1+e_2)b\sin\theta},$$

$$\mu_3=\frac{-(1+e_1)a\sin\theta+(1+e_2)b\cos\theta-b}{-(1+e_1)a\cos\theta-(1+e_2)b\sin\theta+a}.$$

Substituting these values in the reduced Jacobian expression (7)

$$J_4=a(\mu_1-\mu_3)+b(\mu_1\mu_2-2\mu_1\mu_3+\mu_2\mu_3), \tag{10}$$

after some reductions, we get

$$J_4=\frac{2a_1^2b_2(e_2-e_1)\left\{2+e_1+e_2-\dfrac{e_1e_2}{\cos\theta-1}\right\}}{(1+e_2)\left\{b^2(1+e_2)^2\dfrac{\sin^2\theta}{\cos\theta-1}+a_1^2\left[1-\cos\theta+2e_1\cos\theta+\dfrac{e_1^2}{1-\cos\theta}\cos^2\theta\right]\right\}}. \tag{11}$$

Considering e_1 and e_2 as infinitesimals of the first order and $e_2 = \alpha e_1$ (α any finite integer), the limits of $e_1e_2/\cos\theta - 1$, $\sin^2\theta/\cos\theta - 1$, $e_1^2\cos^2\theta/\cos\theta - 1$, for $e_1 \doteq 0$, $\theta \doteq 0$ are finite. According to (6) when none of the derivatives $\phi'(z_i)$, $(i = 1, 2, 3)$ vanish, J_4 is always finite and clearly

$$\lim_{\theta\doteq 0,\, e_1\doteq 0,\, e_2\doteq 0} (J_4) = 0. \tag{12}$$

The denominator of (11) vanishes when any of the derivatives $\phi'(z_i)$, $(i = 1, 2, 3)$ vanish; but in such a case the corresponding derivative may be cancelled from the Jacobian (6), so that on account of (12) the Jacobian (6) always vanishes in case of a combined infinitesimal dilatation and rotation as defined (8) and (9). This is still true in case of either a pure dilatation, or a pure rotation. Conversely, it can be shown without difficulty that it is always possible to determine a dilatation (8) and a rotation (9) in such a manner, that any values of m_1, m_2, m_3, satisfying (6) will be the slopes of the directions of the displacements of A_1, A_2 and A_3.

The condition that three tangents to C at A_1, A_2, A_3 with the slopes m_1, m_2, m_3 are concurrent is

$$a(m_1m_2 - 2m_1m_3 + m_2m_3) + b(m_1 - m_3) = 0. \tag{13}$$

As a and b are positive, the two parentheses in (13) are of opposite sign. Adding the condition $J_4 = 0$, i. e.

$$a(m_1 - m_3) + b(m_1m_2 - 2m_1m_3 + m_2m_3) = 0, \tag{14}$$

(13) and (14) can exist simultaneously only when

$$(m_1m_2 - 2m_1m_3 + m_2m_3)^2 - (m_1 - m_3)^2 = 0. \tag{15}$$

But for $a \neq b$, $|m_1 - m_3| \neq |m_1m_2 - 2m_1m_3 + m_2m_3|$, hence in this case (13) and (14) cannot exist together. When $a = b$, then when (13) is true, (14) is true also. Hence

THEOREM I. *When the Jacobian J_4 (and no others) vanishes for an inscribed rhomb $A_1A_2A_3A_4$, then the tangents to C at A_1, A_2, A_3 are concurrent only when $A_1A_2A_3A_4$ is a square.*

Within the restrictions of this theorem and in all cases where not all four tangents at A_1, A_2, A_3, A_4 are concurrent, the medians M_τ and M_σ associated with the directions of A_1A_3 and A_2A_4 intersect each other singly in the center R of the rhomb (square). In a continuous change of the pair of orthogonal directions (τ, σ) in the neighborhood of A_1A_3 and A_2A_4 the associated inscribed rhombs change continuously. This in connection with the foregoing results leads to

THEOREM II. *If for a certain inscribed rhomb only one of the Jacobians,*

say J_4, vanishes, then the inscribed rhombs in the neighborhood of the given rhomb are continuously connected, and are either infinitesimal dilatations or rotations, or combined infinitesimal dilatations and rotations of the original rhomb, with A_4 stationary.

7. In the system of equations (2) we may also consider z_3, z_2, z_1 successively as the independent variables. Supposing that none of the $\phi'(z_i)$, $i = 1, 2, 3, 4$, vanish at the vertices of the rhomb, which, by properly choosing the coördinate axes, does not imply loss of generality, the vanishing of the corresponding Jacobians is equivalent to

$$J_3 \equiv \begin{vmatrix} 1 & 1 & -1 \\ m_4 & m_1 & -m_2 \\ a & m_1 b & a \end{vmatrix} = 0, \tag{16}$$

or

$$a(2m_1 - m_2 - m_4) + b(m_1 m_2 - m_1 m_4) = 0, \tag{17}$$

$$J_2 \equiv \begin{vmatrix} 1 & 1 & 1 \\ m_3 & m_4 & m_1 \\ -m_3 b & a & m_1 b \end{vmatrix} = 0, \tag{18}$$

or

$$a(m_3 - m_1) + b(m_1 m_4 - 2m_1 m_3 + m_3 m_4) = 0, \tag{19}$$

$$J_1 \equiv \begin{vmatrix} -1 & 1 & 1 \\ -m_2 & m_3 & m_4 \\ a & -m_3 b & a \end{vmatrix} = 0, \tag{20}$$

or

$$a(m_2 + m_4 - 2m_3) + b(m_2 m_3 - m_3 m_4) = 0. \tag{21}$$

The problem now is to find the geometrical meaning of the simultaneous vanishing of two Jacobians, for instance

$$J_4 = 0 \quad \text{and} \quad J_3 = 0.$$

Eliminating a and b between the two equations we find as a necessary condition

$$(m_1 - m_2)[4m_1 m_3 - (m_1 + m_3)(m_2 + m_4)] = 0. \tag{22}$$

The three cases must therefore be considered

$$m_1 = m_2, \tag{1}$$

$$4m_1 m_3 - (m_1 + m_3)(m_2 + m_4) = 0, \tag{2}$$

each of these under the exclusion of the other, and

(3) the existence of (1) and (2) simultaneously.

1. If in $J_4 = 0$ (7) we put $m_2 = m_1$, it reduces to

$$(a + bm_1)(m_1 - m_3) = 0. \tag{23}$$

Similarly $J_3 = 0$ reduces to

$$(a + bm_1)(m_1 - m_4) = 0. \tag{24}$$

As $m_1 = m_2 = m_3 = m_4$ would involve the existence of (2) we exclude this possibility at present. Hence we have as a necessary and sufficient condition for the simultaneous equations $J_4 = 0$, $J_3 = 0$ in case (1)

$$m_1 = m_2 = -\frac{a}{b}. \tag{25}$$

As this condition is otherwise independent of any particular values of m_3 and m_4, it follows that A_3 and A_4 are both stationary and the tangents at A_1 and A_2, Fig. 2, are parallel and are at right angles to the sides A_1A_4

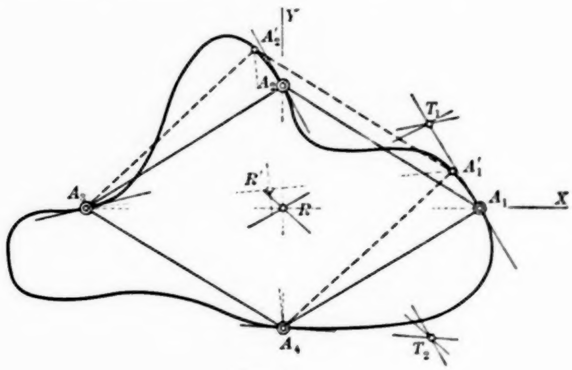


Fig. 2.

and A_2A_3. The tangents T_1R and T_2R to the corresponding medians at R and accordingly these themselves intersect at R. The case where T_1, R, T_2 are collinear will be taken up under (2). We can therefore state

Theorem III. *The rhombs in the neighborhood of $A_1A_2A_3A_4$ in case* (1) *form a continuous set and the transformation corresponding to this set consists of an infinitesimal rotation of A_1 and A_2 about A_4 and A_3 as instantaneous centers.*

(2) $$4m_1m_3 - (m_1 + m_3)(m_2 + m_4) = 0.$$

In this case the tangents at A_1 and A_3 intersect in a point T_1, those at A_2 and A_4 in a point T_2, such that T_1, R, T_2 are collinear, Fig. 3. To prove this, from the equations

$$y = m_1(x - a), \quad y - b = m_2x, \quad y = m_3(x + a), \quad y + b = m_4x$$

of the tangents at A_1, A_2, A_3, A_4, the coordinates of T_1 and T_2 are found for

$$T_1\left\{a\frac{m_1+m_3}{m_1-m_3},\qquad a\frac{2m,\, m_3}{m_1-m_3}\right\}$$

and for

$$T_2\left\{b\frac{2}{m_4-m_2},\qquad b\frac{m_4+m_2}{m_4-m_2}\right\}.$$

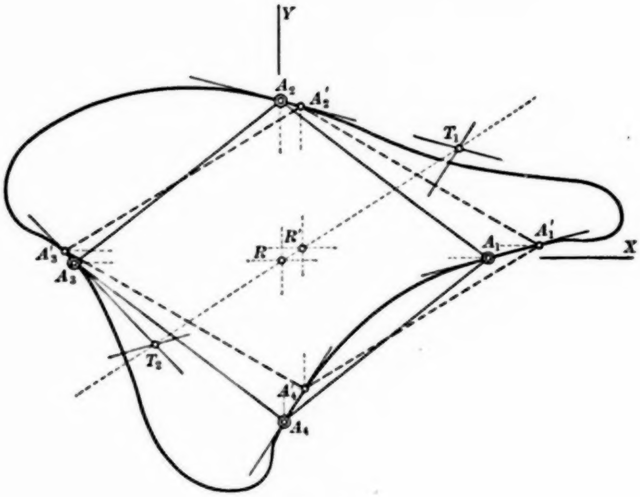


FIG. 3.

The condition for the collinearity of T_1, R, T_2 is

$$\begin{vmatrix} a\dfrac{m_1+m_3}{m_1-m_3} & a\dfrac{2m_1m_3}{m_1-m_3} & 1 \\ b\dfrac{2}{m_4-m_2} & b\dfrac{m_2+m_4}{m_4-m_2} & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0,$$

which reduces to

$$4m_1m_3 - (m_1+m_3)(m_2+m_4) = 0,$$

which clearly is identical with (2). Conversely it follows easily that when the slopes satisfy this condition, the points, T_1, R, T_2 are concurrent.

As T_1R, T_2R are the tangents to the medians through R, *the latter must be tangent at R.* In the continuous changes of the medians in the neighborhood of R, a tangency arises by the coincidence of two points of intersection. *Geometrically case* (2) *is therefore equivalent to the coincidence of two inscribed rhombs with parallel axes, and there exists continuity of connection of the inscribed rhombs in the neighborhood of the given rhomb. The*

infinitesimal transformation consists of a dilatation in the direction of the axes of the rhomb and a translation along the line of collinearity of T_1, R and T_2.

It is of course possible that T_1 and T_2 coincide at a point T. If this point is infinitely distant, then $m_1 = m_2 = m_3 = m_4$; the four tangents are parallel. The coexistence of (1) and (2) is equivalent to two parallel tangents at A_1 and A_2 and collinearity of T_1, R and T_2, so that case (3), except when $m_1 = m_2 = m_3 = m_4$, does not yield anything new.

In a similar manner as in case of $J_4 = 0$, $J_3 = 0$, we find for the simultaneous vanishing of any other two of the four Jacobian expressions, $J_i = 0$ and $J_k = 0$, the necessary condition

$$(26) \qquad (m_j - m_h)[4m_1m_3 - (m_1 + m_3)(m_2 + m_4)] = 0,$$

where m_j and m_h are the slopes of the tangents at the points A_j and A_h which with A_i and A_k form the rhomb.

If in addition to (26) $J_i = 0$, then also $J_k = 0$. Furthermore, the collinearity of T_1, R, T_2 in connections with $J_i = 0$ also makes the three other Jacobian expressions vanish. Hence the

THEOREM IV: *If the points T_1, R, T_2 are collinear, in other words, if the two medians associated with the directions parallel to the axes of a rhomb inscribed to an ordinary closed curve touch each other at the center of the rhomb, then all Jacobians defined in connection with the system of functions* (2) *vanish simultaneously with the system if one of the Jacobians vanishes.*

8. In general, when no particular assumptions about the functions $\phi(t)$ and $\psi(t)$ are made, the system of equations (2) and any of the Jacobians, like (4), vanish simultaneously for a finite number of sets of values z_1, z_2, z_3, z_4 and if no other Jacobian vanishes for any of these sets, then there exists for every corresponding rhomb the condition stated in theorem II. If in addition another Jacobian vanishes, then all other Jacobians vanish, when the points T_1, R, T_2 are collinear, as stated in theorem IV. When only two Jacobians vanish simultaneously with (2), theorem III results. If for an inscribed rhomb the points T_1, R, T_2 are collinear and none of the Jacobians vanish, then the rhombs in the neighborhood of the given rhomb are also continuously connected. This corresponds to the case in which two points of intersection of the medians associated with two orthogonal continuously changing directions coincide.

UNIVERSITY OF ILLINOIS,
May, 1913.

ON THE IRREGULAR INTEGRALS OF LINEAR DIFFERENTIAL EQUATIONS.*

By Clyde E. Love.

1. In the present paper it is proposed to study the integrals of a homogeneous linear differential equation in the neighborhood of an irregular point, particularly with a view to determining the existence and form of their asymptotic developments.† The existence of such developments is well-established provided the roots of the so-called characteristic equation are all distinct,‡ but for the case of multiple roots no general discussion has appeared. In the following work, while the roots are unrestricted as to their order of multiplicity, a certain limitation is placed upon them for the purpose of reducing the algebraic difficulties of the analysis. This limitation is of such a nature that the case of distinct roots is included as a special case of the problem here treated.

The irregular point is taken as the point at infinity, and the independent variable is restricted to real and positive values.

2. As a preliminary step we shall formulate two general theorems§ on linear differential equations, which will form the basis for the subsequent investigation.

Take for consideration the differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0, \tag{1}$$

whose coefficients $a_1(x)$, $a_2(x)$, $\cdots$, $a_n(x)$ are continuous and possess n continuous derivatives for all sufficiently large positive values of x. Let $z_1, z_2, \cdots, z_n$ denote n auxiliary functions of x which, together with their first n derivatives, are likewise continuous when x is large and positive, and are such that the determinant

$$Q(x) = \begin{vmatrix} z_1 & z_1' & \cdots & z_1^{(n-1)} \\ z_2 & z_2' & \cdots & z_2^{(n-1)} \\ \vdots & & & \\ z_n & z_n' & \cdots & z_n^{(n-1)} \end{vmatrix} \tag{2}$$

never vanishes. Let $A_r(x)$ be the minor of $Q(x)$ with respect to the element

* Read before the American Mathematical Society, Dec. 27, 1913.

† In Poincaré's sense: cf. Acta Mathematica, vol. 8 (1886), p. 297.

‡ Cf., for example, Horn, Journal für Mathematik, vol. 138 (1910), pp. 159–191.

§ Cf. Dini, Annali di Matematica, Ser. III, vol. 2 (1899), pp. 297–324.

$z_r^{(n-1)}$. Also define

$$(3)\qquad Z_r(x) = z_r a_n - (z_r a_{n-1})' + \cdots + (-1)^{n-1}(z_r a_1)^{(n-1)} + (-1)^n z_r^{(n)}, \qquad r = 1, 2, \cdots, n,$$

and denote by $q(x, x_1)$ the determinant formed from $Q(x)$ by replacing the element $z_r^{(n-1)}(x)$ by $Z_r(x_1)$, $r = 1, \cdots, n$.

Place

$$(4)\qquad K(x, x_1) = \frac{(-1)^{n-1}q(x, x_1)}{Q(x)};$$

$$(5)\qquad g_r(x) = u_{r,0}(x) = v_{r,0}(x) = \frac{(-1)^{n-1}C_r A_r(x)}{Q(x)}, \quad r = 1, \cdots n,$$

where C_r is an arbitrary constant;

$$u_{r,\lambda}(x) = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{\lambda-1}} K(x, x_1)K(x_1, x_2) \cdots K(x_{\lambda-1}, x_\lambda)g_r(x_\lambda)dx_\lambda \cdots dx_2 dx_1, \quad \lambda = 1, 2, \cdots,$$

$$v_{r,\lambda}(x) = \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{\lambda-1}}^\infty K(x, x_1)K(x_1, x_2) \cdots K(x_{\lambda-1}, x_\lambda)g_r(x_\lambda)dx_\lambda \cdots dx_2 dx_1, \quad \lambda = 1, 2, \cdots.$$

Then we have

THEOREM A: *Suppose that for all values of x greater than some constant, the series*

$$(6)\qquad y_r(x) = \sum_{\lambda=0}^{\infty} v_{r,\lambda}(x)$$

satisfies the following conditions: (*a*) *the series converges*; (*b*) *the series for $y_r(x_1)$ when multiplied by $K(x, x_1)$ may be integrated term by term with respect to x_1 from x to ∞*; (*c*) *the series defines a function $y_r(x)$ such that each of the integrals*

$$\int_x^\infty y_r(x)Z_s(x)dx, \qquad s = 1, \cdots, n,$$

has a meaning. Then for such values of x the function $y_r(x)$ is an integral of (1).

THEOREM B: *Suppose that a constant a can be found such that for all values of $x > a$ the series*

$$y_r(x) = \sum_{\lambda=0}^{\infty} u_{r,\lambda}(x)$$

satisfies the following conditions: (*a*) *the series converges*; (*b*) *the series for $y_r(x_1)$ when multiplied by $K(x, x_1)$ may be integrated term by term with respect*

to x_1 from a to x. Then for such values of x the function $y_r(x)$ is an integral of (1).

To prove Theorem A, place

$$(7)\quad \begin{cases} p_{s,\,0} &= z_s, \\ p_{s,\,1} &= z_s a_1 - z_s' = z_s a_1 - p_{s,\,0}', \\ \quad\vdots & \\ p_{s,\,n-1} &= z_s a_{n-1} - p_{s,\,n-2}'; \qquad s = 1, \cdots, n; \end{cases}$$

$$\Phi_s(x) = \int_x^\infty y_r Z_s dx, \qquad s = 1, \cdots, r-1, r+1, \cdots, n;$$

$$\Phi_r(x) = \int_x^\infty y_r Z_r dx + C_r;$$

$$\Delta_r(x) = \begin{vmatrix} p_{1,\,0} & p_{1,\,1} & \cdots & p_{1,\,n-2} & \Phi_1(x) \\ \vdots & & & & \\ p_{n,\,0} & p_{n,\,1} & \cdots & p_{n,\,n-2} & \Phi_n(x) \end{vmatrix},$$

$$\Delta(x) = \begin{vmatrix} p_{1,\,0} & p_{1,\,1} & \cdots & p_{1,\,n-1} \\ \vdots & & & \\ p_{n,\,0} & p_{n,\,1} & \cdots & p_{n,\,n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} Q(x).$$

Now by condition (b) equation (6) may be written

$$(8)\qquad y_r(x) = g_r(x) + \int_x^\infty y_r(x_1) K(x, x_1) dx_1.$$

By substituting the values of $g_r(x)$ and $K(x, x_1)$ in (8) we find

$$y_r(x) = \Delta_r(x)/\Delta(x),$$

so that it suffices for our proof to show that this function is an integral of (1).

To do this, consider the system of n functions $\eta_0, \eta_1, \cdots, \eta_{n-1}$ each defined for all values of x sufficiently large by means of the following system of n linear equations:

$$(9)\qquad p_{s,\,0}\eta_{n-1} + p_{s,\,1}\eta_{n-2} + \cdots + p_{s,\,n-1}\eta_0 = \Phi_s(x), \quad s = 1, \cdots, n.$$

We have at once

$$(10)\qquad \eta_0 = \frac{\Delta_r(x)}{\Delta(x)} = y_r.$$

Upon differentiating equations (9) with respect to x, and making use of (7) and (10), we find

$$(11)\qquad z_s\theta + p_{s,\,0}'\theta_1 + p_{s,\,1}'\theta_2 + \cdots + p_{s,\,n-2}'\theta_{n-1} = 0, \quad s = 1, \cdots, n,$$

where

(12) $$\theta = \eta_{n-1}' + a_1(x)\eta_{n-2}' + \cdots + a_{n-1}(x)\eta_0' + a_n(x)\eta_0,$$

(13) $$\begin{cases} \theta_1 = \eta_{n-1} - \eta_{n-2}', \\ \theta_2 = \eta_{n-2} - \eta_{n-3}', \\ \vdots \\ \theta_{n-1} = \eta_1 - \eta_0'. \end{cases}$$

The system (11) consists of n homogeneous linear equations in the n quantities $\theta, \theta_1, \cdots, \theta_{n-1}$. By virtue of the relation

$$p_{t,s}' = z_t a_{s+1} - p_{t,s+1}, \qquad t = 1, \cdots, n; \qquad s = 0, 1, \cdots, n-2,$$

the discriminant of the system reduces at once to $(-1)^{n-1}Q(x)$, and hence by our hypotheses does not vanish for any value of x under consideration. Whence

$$\theta \equiv \theta_1 \equiv \cdots \equiv \theta_{n-1} \equiv 0,$$

or by (10) and (13),

$$\eta_s = y_r^{(s)}, \qquad s = 0, 1, \cdots, n.$$

Substituting in (12), we find

$$y_r^{(n)} + a_1(x)y_r^{(n-1)} + \cdots + a_n(x)y_r \equiv 0,$$

which was to be proved.

The proof of Theorem B follows similar lines, and may be omitted.

These theorems differ only slightly from certain results due to Dini.* He was first led by synthetic processes to the converse theorems, and was then able to establish the direct theorems by reversing his previous reasoning. The appearance of artificiality in the proof as here outlined is chiefly due to the fact that only this reversed line of argument is presented.

3. In the differential equation (1), suppose that the coefficients $a_1(x)$, $a_2(x), \cdots, a_n(x)$ are real or complex functions developable, for large real positive values of x, in asymptotic (or convergent) power series of the form

$$a_r(x) \sim x^{rk}\left[a_{r,0} + \frac{a_{r,1}}{x} + \cdots\right], \qquad r = 1, \cdots, n,$$

k being 0 or a positive integer. Suppose also that the first $n - r$ derivatives of $a_r(x)$ possess asymptotic developments in the same x-region.

We proceed to apply the theorems of § 2 to the problem of finding asymptotic solutions of (1), valid for large real values of x.

Suppose that the characteristic equation

$$\phi(m) = m^n + a_{1,0}m^{n-1} + \cdots + a_{n,0} = 0$$

corresponding to (1) has l different roots $m_1, m_2, \cdots, m_l$, occurring $n_1, n_2, \cdots, n_l$ times respectively: $(n_1 + n_2 + \cdots + n_l = n)$.

* Loc. cit.

Let us choose n auxiliary functions (cf. § 2) of the form

(14) $$z_{n_1+\cdots+n_{r-1}+q} = x^{-\eta_{r,q}} e^{-f_{r,q}(x)+h_{r,q}(x)}, \qquad r = 1, \cdots, l;\ q = 1, \cdots, n_r,$$

where

$$f_{r,q}(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,q} x^{k+1-\frac{1}{n_r}}}{k+1-\frac{1}{n_r}} + \frac{\beta_{r,q} x^{k+1-\frac{2}{n_r}}}{k+1-\frac{2}{n_r}} + \cdots + \frac{\zeta_{r,q} x^{\frac{1}{n_r}}}{\frac{1}{n_r}}$$

and

$$h_{r,q}(x) = \frac{\theta_{r,q} x^{-\frac{1}{n_r}}}{\frac{1}{n_r}} + \frac{\kappa_{r,q} x^{-\frac{2}{n_r}}}{\frac{2}{n_r}} + \cdots + \frac{\nu_{r,q} x^{-s+1}}{s-1},$$

$\alpha_{r,q}, \beta_{r,q}, \cdots, \nu_{r,q}$ being undetermined constants and s an arbitrary positive integer.

For brevity let us write

$$z_{n_1+\cdots+n_{r-1}+q} = z_{r,q},$$

$$Z_{n_1+\cdots+n_{r-1}+q} = Z_{r,q},$$

$$g_{n_1+\cdots+n_{r-1}+q} = g_{r,q}.$$

Now upon forming $Z_{r,q}(x)$ as given by (3), we find that

(15) $$Z_{r,q}(x) = (-1)^n z_{r,q} x^{nk}[S_{r,q}(x) + x^{-s-k} P_{1,r,q}(x^{1/n_r})],$$

where* $S_{r,q}(x)$ is a polynomial of degree $n_r(s+k) - 1$ in x^{-1/n_r}. In $S_{r,q}(x)$, the coefficients of x^{-j/n_r}, $j = 0, 1, \cdots, n_r - 1$, vanish identically because of the fact that

$$\varphi(m_r) = \varphi'(m_r) = \cdots = \varphi^{(n_r-1)}(m_r) = 0.$$

Further, writing this polynomial in the form

$$S_{r,q}(x) = x^{-1}[\mathrm{A}_{r,q} + \mathrm{B}_{r,q} x^{-1/n_r} + \Gamma_{r,q} x^{-2/n_r} + \cdots + \mathrm{N}_{r,q} x^{-s-k+1/n_r}],$$

we have

$$\mathrm{A}_{r,q} = \frac{\varphi^{(n_r)}(m_r)}{n_r!} \alpha_{r,q}{}^{n_r} + \psi(m_r),$$

where

$$\psi(m) = a_{1,1} m^{n-1} + a_{2,1} m^{n-2} + \cdots + a_{n,1};$$

* Throughout the work, the symbol $P(x)$ will denote a function expressible, for large values of x, in the form

$$P(x) = a_0 + \frac{a_1}{x} + \cdots + \frac{a_p + \epsilon_p(x)}{x^p}, \quad \lim_{x=\infty} \epsilon_p(x) = 0,$$

where p is an arbitrary positive integer.

and

$$\mathrm{B}_{r,q} = \frac{\varphi^{(n_r)}(m_r)}{(n_r - 1)!}\alpha_{r,q}{}^{n_r-1}\beta_{r,q} + \sigma_{r,q}(m_r, \alpha_{r,q}),$$

$$\Gamma_{r,q} = \frac{\varphi^{(n_r)}(m_r)}{(n_r - 1)!}\alpha_{r,q}{}^{n_r-1}\gamma_{r,q} + \tau_{r,q}(m_r, \alpha_{r,q}, \beta_{r,q}),$$

$$\cdot$$

$$\mathrm{N}_{r,q} = \frac{\varphi^{(n_r)}(m_r)}{(n_r - 1)!}\alpha_{r,q}{}^{n_r-1}\nu_{r,q} + \omega_{r,q}(m_r, \alpha_{r,q}, \cdots, \mu_{r,q}),$$

where $\sigma_{r,q}, \tau_{r,q}, \cdots, \omega_{r,q}$ are certain polynomials in the indicated arguments, which it would be useless to write out in full.

Let us now try to determine the coefficients $\alpha_{r,q}, \beta_{r,q}, \cdots, \nu_{r,q}$ by placing*

(16) $$\mathrm{A}_{r,q} = \mathrm{B}_{r,q} = \cdots = \mathrm{N}_{r,q} = 0.$$

For simplicity we shall assume that *no multiple root of the characteristic equation is also a root of the equation* $\psi(m) = 0$, or in other words that there is no value of m satisfying simultaneously the three equations†

$$\varphi(m) = \varphi'(m) = \psi(m) = 0.$$

Under this hypothesis equations (16) can always be solved, and they serve to determine n definite, distinct functions $z_{r,q}$ of the form (14).

4. The quantities $R[f_{r,q}(x)]$, where $R[x]$ denotes the real part of x, will play an important part in what follows. We consider first the case in which these real parts are all distinct, and shall suppose for definiteness that when x is sufficiently large

(17) $$R[f_{1,1}(x)] < R[f_{1,2}(x)] < \cdots < R[f_{1,n_1}(x)] \\ < R[f_{2,1}(x)] < \cdots < R[f_{l,n_l}(x)].$$

The argument to be used will apply with changes only in notation if these real parts occur in any other order when arranged with respect to their magnitude, so that the assumption (17) involves no loss of generality.

Let n' denote the least common multiple of $n_1, n_2, \cdots, n_l$, and set

$$f(x) = f_{1,1}(x) + f_{1,2}(x) + \cdots + f_{l,n_l}(x),$$

$$\eta = \eta_{1,1} + \eta_{1,2} + \cdots + \eta_{l,n_l} - \frac{(n-1)nk - n + l}{2},$$

$$\Delta = (-1)^{\frac{n(n-1)}{2}}\Pi(m_r - m_s)^{n_r n_s}\Pi(\alpha_{r,q} - \alpha_{r,p}),$$

* As soon as the constants $\eta_{r,q}$ have been determined, s is to be selected so as to satisfy the inequality (21).

† It is understood, of course, that the case $\psi(m) \equiv 0$ is also excluded, except when the roots of the characteristic equation are all distinct.

where the first product is composed of all the factors $m_r - m_s$ that can be formed with $r = 2, 3, \cdots l$ and $r > s$; the second consists of all factors $\alpha_{r,q} - \alpha_{r,p}$ that can be formed with $r = 1, 2, \cdots, l$; $q = 2, 3, \cdots, n_r$ and $q > p$.

With this notation we find, by (2),

$$Q(x) = \Delta e^{-f(x)} x^{-\eta} P_2(x^{1/n'}),$$

where the constant term in* $P_2(x^{1/n'})$ is unity, so that the hypothesis $Q(x) \neq 0$ for all values of x under consideration is satisfied. Further, with proper choice of the arbitrary constant $C_{r,q}$, we may write by (5),

$$g_{r,q}(x) = e^{f_{r,q}(x)} x^{\rho_{r,q}} P_{3,r,q}(x^{1/n'}), \tag{18}$$

where we have set

$$\rho_{r,q} = \eta_{r,q} + 1 - (n-1)k,$$

and where the constant term in $P_{3,r,q}(x^{1/n'})$ is unity. And by (4) we have

$$K(x, x_1) = \sum_{r=1}^{l} \sum_{q=1}^{n_r} e^{f_{r,q}(x)} x^{\rho_{r,q}} P_{3,r,q}(x^{1/n'}) e^{-f_{r,q}(x_1)} x_1^{-\rho_{r,q}-s} P_{4,r,q}(x_1^{1/n'}). \tag{19}$$

5. We shall now write out by Theorem A a trial solution of (1) corresponding to that one of the functions $g_{r,q}(x)$ whose exponential factor has the least real part, which by (17) is $g_{1,1}(x)$, and show that the conditions of the theorem are satisfied. We have

$$y_{1,1} = g_{1,1}(x) + \sum_{\lambda=1}^{\infty} v_{1,1,\lambda}(x), \tag{20}$$

where

$$v_{1,1,\lambda}(x) = \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{\lambda-1}}^\infty K(x, x_1) K(x_1, x_2) \cdots K(x_{\lambda-1}, x_\lambda) g_{1,1}(x_\lambda) dx_\lambda \cdots dx_2 dx_1.$$

Placing

$$\bar{g}_{r,q}(x) = e^{f_{r,q}(x)} x^{\rho_{r,q}},$$

$$\bar{K}(x, x_1) = \sum_{r=1}^{l} \sum_{q=1}^{n_r} e^{f_{r,q}(x)} x^{\rho_{r,q}} e^{-f_{r,q}(x_1)} x_1^{-\rho_{r,q}-s},$$

$$\bar{v}_{r,q,\lambda}(x) = \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{\lambda-1}}^\infty \bar{K}(x, x_1) \bar{K}(x_1, x_2) \cdots \bar{K}(x_{\lambda-1}, x_\lambda) \bar{g}_{1,1}(x_\lambda) dx_\lambda \cdots dx_2 dx_1,$$

let us study the behavior, for large values of x, of the function $\bar{v}_{1,1,\lambda}(x)$.

We first choose an arbitrary positive integer p, and then take s so large

* Cf. footnote, p. 149.

that*

$$s > p + 2 + 2\bar{\rho}, \tag{21}$$

where $\bar{\rho}$ is a positive number at least as great as any of the quantities

$$|R[\rho_{r,\,q}]|.$$

Since, by (17),

$$R[f_{1,\,1}(x_\lambda) - f_{r,\,q}(x_\lambda)] \leqq 0$$

for all values of r and q, each of the functions

$$|e^{-f_{r,\,q}(x_\lambda)}x_\lambda^{-\rho_{r,\,q}-s}e^{f_{1,\,1}(x_\lambda)}x_\lambda^{\rho_{1,\,1}}|$$

decreases monotonically for all values of x_λ under consideration, and we have

$$|\bar{K}(x_{\lambda-1}, x_\lambda)\bar{g}_{1,\,1}(x_\lambda)| < n\,|e^{f_{1,\,1}(x_{\lambda-1})}x_{\lambda-1}^{\rho_{1,\,1}}|x_\lambda^{-p-2}.$$

Similarly,

$$|\bar{K}(x_{\lambda-2}, x_{\lambda-1})\bar{K}(x_{\lambda-1}, x_\lambda)g_{1,\,1}(x_\lambda)| < n^2|e^{f_{1,\,1}(x_{\lambda-2})}x_{\lambda-2}^{\rho_{1,\,1}}|x_{\lambda-1}^{-p-2}x_\lambda^{-p-2}.$$

Proceeding in this way, we find

$$\begin{aligned}|\bar{v}_{1,\,1,}\ (x)| < n^\lambda|e^{f_{1,\,1}(x)}x^{\rho_{1,\,1}}| \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{\lambda-1}}^\infty (x_1x_2\cdots x_\lambda)^{-p-2}\,dx_\lambda \\ \cdots dx_2dx_1 < |e^{f_{1,\,1}(x)}x^{\rho_{1,\,1}}|\frac{n^\lambda}{(p+1)^\lambda}\cdot\frac{1}{x^{\lambda(p+1)}}.\end{aligned} \tag{22}$$

By virtue of the defining property of the functions $P(x)$ (see footnote, p. 149), we may take x so large that

$$|v_{1,\,1,\,\lambda}(x)| < 2|\bar{v}_{1,\,1,\,\lambda}(x)|.$$

Whence, by (22) the series $\sum_{\lambda=1}^\infty v_{1,\,1,\,\lambda}(x)$ converges absolutely and uniformly for all sufficiently large values of x, so that condition (a) of Theorem A is satisfied. Further, by comparison with the geometric series whose general term is twice the right member of (22), it appears that

$$\begin{aligned}\sum_{\lambda=1}^\infty|v_{1,\,1,\,\lambda}(x)| < |e^{f_{1,1}(x)}x^{\rho_{1,1}}|\frac{2n}{p+1}\cdot\frac{1}{x^{p+1}}\cdot\frac{1}{1-\dfrac{n}{(p+1)x^{p+1}}} \\ < |e^{f_{1,1}(x)}x^{\rho_{1,1}}|\frac{4n}{p+1}\cdot\frac{1}{x^{p+1}}.\end{aligned} \tag{23}$$

That condition (b) is satisfied follows from the results just established, together with the following theorem:†

* Cf. footnote, p. 150.

† Bromwich, Infinite Series, p. 453.

If $\Sigma f_n(x)$ converges uniformly in any fixed interval $a \leqq x \leqq b$, where b is arbitrary, and if $\varphi(x)$ is continuous for all finite values of x, then

$$\int_a^\infty \varphi(x)\Sigma f_n(x)dx = \Sigma \int_a^\infty \varphi(x)f_n(x)dx,$$

provided that the integral $\int_a^\infty |\varphi(x)|\Sigma|f_n(x)|dx$ is convergent.

For, if we take $a = x$, $x = x_1$, $\varphi(x_1) = K(x, x_1)$, $f_n(x_1) = v_{1,1,\lambda}(x_1)$, it only remains to show that $\int_x^\infty |K(x, x_1)| \sum_{\lambda=0}^{\infty} |v_{1,1,\lambda}(x_1)|dx_1$ converges. But by (4) and (22) this breaks up into n integrals whose convergence is assured by that of the integrals $\int_x^\infty |e^{f_{1,1}(x_1)-f_{r,q}(x_1)}x_1^{\rho_{1,1}-\rho_{r,q}-\epsilon}|dx_1$, and each of these converges by reason of (17) and (21).

These same inequalities are sufficient to show that condition (c) holds, if it be recalled that, in (15), we have made

$$S_{r,q}(x) \equiv 0.$$

Hence $y_{1,1}$ as given by (20) is an actual solution of (1).

6. When we attempt to write out a particular integral of (1) corresponding to any other of the functions $g_{r,q}(x)$, the above process fails for the reason that $R[f_{r,q}(x) - f_{1,1}(x)] > 0$ and some of the integrals in $v_{r,q,\lambda}(x)$ diverge. To avoid this difficulty we shall first modify slightly our system of auxiliary functions and then use Theorem B.

The adjoint equation corresponding to (1), viz.

$$z^{(n)} - (a_1z)^{(n-1)} + \cdots + (-1)^n za_n = 0, \tag{24}$$

satisfies all the conditions that have been imposed upon (1); it has therefore a particular integral given by (20). The functions $f_{r,q}(x)$ for (24) are obviously the same as for (1), except that all signs are changed. The one having the least real part is therefore $-f_{l,n_l}(x)$, and the corresponding solution of (24) is found by direct computation to be

$$\bar{z}_{l,n_l}x = e^{-f_{l,n_l}(x)}x^{-\eta_{l,n_l}}P_{5,l,n}(x^{1/n}).$$

In our system of auxiliary functions let us replace z_{l,n_l} by this function $\bar{z}_{l,n_l}$, thus making $Z_{l,n_l} \equiv 0$. This leaves $Q(x)$, $g_{r,q}(x)$ and $\bar{g}_{r,q}(x)$ unchanged in essentials, while in $K(x, x_1)$ and $\bar{K}(x, x_1)$ the term for which $r = l$ and $q = n_l$ no longer appears. Now write by Theorem B a trial integral of (1) of the form

$$y_{l,n_l} = g_{l,n_l}(x) + \sum_{\lambda=1}^{\infty} u_{l,n_l,\lambda}(x). \tag{25}$$

Put

$$\bar{u}_{r,\,q,\,\lambda}(x) = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{\lambda-1}} \bar{K}(x, x_1)\bar{K}(x_1, x_2) \cdots \bar{K}(x_{\lambda-1}, x_\lambda)\bar{g}_{r,\,q}(x_\lambda)dx_\lambda \cdots dx_2 dx_1.$$

Now we have

$$R[f_{l,\,n_l}(x_\lambda) - f_{r,\,q}(x_\lambda)] > 0$$

for all values of r and q occurring in $\bar{K}(x_{\lambda-1}, x_\lambda)$. Thus each of the functions $|e^{f_{l,\,n_l}(x_\lambda) - f_{r,\,q}(x_\lambda)} x_\lambda^{\rho_{l,\,n_l} - \rho_{r,\,q} - s}|$ increases monotonically throughout the interval $a \leqq x_\lambda \leqq x_{\lambda-1}$, so that

$$|\bar{K}(x_{\lambda-1}, x_\lambda)\bar{g}_{l,\,n_l}(x_\lambda)| < n\,|e^{f_{l,\,n_l}(x_{\lambda-1})} x_{\lambda-1}{}^{\rho_{l,\,n_l} - (p+2)}|.$$

Similarly,

$$|\bar{K}(x_{\lambda-2}, x_{\lambda-1})\bar{K}(x_{\lambda-1}, x_\lambda)\bar{g}_{l,\,n_l}(x_\lambda)| < n^2 |e^{f_{l,\,n_l}(x_{\lambda-2})} x_{\lambda-1}{}^{\rho_{l,\,n_l} - 2(p+2)}|.$$

And

$$|\bar{u}_{r,\,q,\,\lambda}(x)| < n^\lambda |e^{f_{l,\,n_l}(x)} x^{\rho_{l,\,n_l}}|\, x^{-\lambda(p+2)} \int_a^x \int_a^{x_1} \cdots \int_a^{x_{\lambda-1}} dx_\lambda \cdots dx_2 dx_1$$

$$< n^\lambda |e^{f_{l,\,n_l}(x)} x^{\rho_{l,\,n_l}}|\, x^{-\lambda(p+1)}.$$

Whence by reasoning similar to that used in the previous case it appears that the conditions of the theorem are satisfied, and $y_{l,\,n_l}$ is a solution of (1).

7. We may now form by (25) a second solution of (24), which takes the form

$$\bar{z}_{1,\,1} = e^{-f_{1,\,1}(x)} x^{-\eta_{1,\,1}} P_{5,\,1,\,1}(x^{1/n_1}).$$

This being introduced into the auxiliary system in place of $z_{1,\,1}$, the term in $K(x, x_1)$ and $\bar{K}(x, x_1)$ for which $r = 1$ and $q = 1$ disappears, since now $Z_{1,\,1} \equiv 0$.

Upon writing out by Theorem A the function*

$$y_{1,\,2} = g_{1,\,2}(x) + \sum_{\lambda=1}^{\infty} v_{1,\,2,\,\lambda}(x),$$

the difficulty which formerly presented itself is seen to be obviated by our more favorable choice of auxiliary functions, since for all values of r and q now occurring

$$R[f_{1,\,2}(x) - f_{r,\,q}(x)] \leqq 0.$$

Having proved by the reasoning of § 5 that $y_{1,\,2}$ is an integral of (1), we need only introduce a solution $\bar{z}_{l,\,n_l-1}$ of (24) in place of $z_{l,\,n_l-1}$ to be able to obtain by Theorem B another solution of (1).

Continuing in this way, we get n integrals of (1) having one or the other

* Of course if $n_1 = 1$, we should here form $y_{2,1}$.

of the forms

$$(26)\qquad \begin{cases} y_{r,\,q} = g_{r,\,q}(x) + \sum_{\lambda=1}^{\infty} u_{r,\,q,\,\lambda}(x), \\ y_{r,\,q} = g_{r,\,q}(x) + \sum_{\lambda=1}^{\infty} v_{r,\,q,\,\lambda}(x). \end{cases}$$

It is easily shown, and is in fact almost self-evident, that these integrals are linearly independent.

8. In case the quantities $R[f_{r,\,q}(x)]$, while all distinct, occur in another order than that of (17), the above argument evidently serves without modification, merely giving us the integrals in a different order.

In case the real parts are not all distinct, we can obtain at once by Theorem A particular integrals corresponding to each of the functions $f_{r,\,q}(x)$ whose real parts are equal and least. A group of solutions of (24) can then be formed, after which Theorem B serves to determine a group of solutions of (1) corresponding to those functions $f_{r,\,q}(x)$ whose real parts are equal and greatest. The rest of the process is obvious.

In particular, if the quantities $R[f_{r,\,q}(x)]$ are all equal, the complete system of integrals can be written down by Theorem A, at once.

9. It is an immediate consequence of (23) and the similar result for the other solutions (26) that

$$\lim_{x=\infty} x^p \left[\frac{y_{r,\,q} - g_{r,\,q}(x)}{g_{r,\,q}(x)} \right] = 0, \quad r = 1, \cdots, l;\ q = 1, \cdots, n_r.$$

Whence

$$y_{r,\,q} \sim g_{r,\,q}(x).$$

By direct substitution in the differential equation it appears that we may write

$$y_{r,\,q} \sim e^{f_{r,\,q}(x)} x^{\rho_{r,\,q}} \sum_{i=0}^{n_r-1} \sum_{j=0}^{\infty} \frac{A_{r,\,q,\,i,\,j}}{x^{j+i/n_r}},$$

where the $A_{r,\,q,\,i,\,j}$ are determinate constants and $A_{r,\,q,\,0,\,0} = 1$.

10. In summary, we may state the following result:

Suppose that the differential equation

$$(1)\qquad y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = 0$$

satisfies the following conditions for all sufficiently large real positive values of x:

(*a*) *The coefficients* $a_1(x), \cdots, a_n(x)$ *and their first* n *derivatives are continuous.*

(*b*) *The same coefficients are developable in asymptotic* (*or convergent*) *series of the form*

$$a_r(x) \sim x^{rk} \left[a_{r,\,0} + \frac{a_{r,\,1}}{x} + \cdots \right], \qquad r = 1, \cdots, n,$$

where k is 0 or a positive integer, and the first $n - r$ derivatives of $a_r(x)$ also possess asymptotic developments.

(*c*) *The characteristic equation*

$$m^n + a_{1,0}m^{n-1} + \cdots + a_{n,0} = 0$$

has l roots $m_1, \cdots, m_l$, occurring respectively $n_1, \cdots, n_l$ times: $(n_1 + \cdots + n_l = n)$ and such that no multiple root of the characteristic equation is also a root of the equation

$$a_{1,1}m^{n-1} + a_{2,1}m^{n-2} + \cdots + a_{n,1} = 0.$$

Then equation (1) *possesses a fundamental system of solutions $y_{r,q}$ $(r = 1, \cdots, l;\ q = 1, \cdots, n_r)$ developable asymptotically in the form*

$$y_{r,q} \sim e^{f_{r,q}(x)} \sum_{i=0}^{n_r-1} x^{\rho_{r,q}-i/n_r} \sum_{j=0}^{\infty} \frac{A_{r,q,i,j}}{x^j},$$

where $f_{r,q}(x)$ is a certain polynomial of degree $n_r(k + 1)$ in x^{1/n_r}, the quantities $\rho_{r,q}$ and $A_{r,q,i,j}$ are determinate constants, and $A_{r,q,0,0} = 1$.

University of Michigan.

SOME SOLUTIONS OF THE PELLIAN EQUATIONS $x^2 - Ay^2 = \pm 4$.

By E. E. Whitford.

The Pellian equations $x^2 - Ay^2 = \pm 4$, as well as the equations $x^2 - Ay^2 = \pm 1$, are of great importance and interest in the theory of numbers, and in particular in determining the units of a real quadratic domain.

The units of a quadratic domain are those integers of the domain which divide every integer of the domain. For an imaginary quadratic domain the number of integers is limited. The domain of the square root of negative one, $k(i)$, has four units, ± 1, $\pm i$; $k(\sqrt{-3})$ has six units, ± 1, $\pm (1 \pm \sqrt{-3})/2$, and all others have only two units, ± 1. But a real quadratic domain has an infinite number of units. It becomes convenient to distinguish a fundamental unit; it is the smallest unit of the domain > 1.

Now the solutions of the Pellian equations $x^2 - Ay^2 = \pm 1$ or ± 4 determine units for the domain $k(\sqrt{A})$ but the fundamental solution of the equation $x^2 - Ay^2 = 1$ does not determine the fundamental unit when the solution of the equations $x^2 - Ay^2 = -1$ or 4 or -4 is possible. While the solution of the equation $x^2 - Ay^2 = 1$ is always possible for every non-square positive integral value of A, the solution of the three other equations is not always possible for every such value of A. A necessary condition for the solution of the equations $x^2 - Ay^2 = \pm 4$ with x, y, not both even is that $A \equiv 5 \pmod 8$.

To illustrate, to obtain a unit of the domain $k(\sqrt{69})$ we might solve the Pell equation $x^2 - 69y^2 = 1$, obtaining for the smallest values, greater than 0, for $x, y, x = 7{,}775$, $y = 936$, and hence for one unit $7{,}775 + 936\sqrt{69}$. But to obtain the fundamental unit solve the equation $x^2 - 69y^2 = +4$, since this equation has a solution, and for the smallest values of x, y, get $x = 25$, $y = 3$; and the fundamental unit for the domain $k(\sqrt{69})$ is $(25 + 3\sqrt{69})/2$.

The fundamental solutions* for the Pell equation have been published up to $A = 1{,}700$; and of the equation $x^2 - Ay^2 = 4$ or the equation $x^2 - Ay^2 = -4$ up to $A = 997$.

* For account of the solutions of the Pell equations $x^2 - Ay^2 = \pm 1$, see E. E. Whitford, "The Pell Equation," New York, 1912; and for solutions of the equations $x^2 - Ay^2 = \pm 4$, see A. Cayley, "Note sur l'équation $x^2 - Dy^2 = \pm 4$, $D \equiv 5 \pmod 8$," Journal für die reine und angewandte Mathematik, vol. 53 (1857), p. 369.

The following table gives the fundamental solutions of the equations $x^2 - Ay^2 = \pm 4$ for $A \equiv 5 \pmod 8$ from $A = 1{,}005$ to $1{,}997$, where such solutions are possible. Where the solution of $x^2 - Ay^2 = -4$ is possible, that solution is given first, followed by the solution of the equation $x^2 - Ay^2 = 4$.

Table of the Fundamental Solutions of the Equations $\mathbf{x^2 - Ay^2 = \pm 4}$, $\mathbf{A \equiv 5 \pmod 8}$ where such Solutions are Possible, from A = 1005 to A = 1997.

A	x	y
1,005	1,807	57
1,013	923	29
	851,931	26,767
1,021	85 745,895	2 683,493
	7,352 358,509 351,027	230 098,509 011,235
1,029	57,965	1,807
1,037	161	5
	25,923	805
1,045	97	3
1,061	264,395	8,117
	69,904 716,027	2,146 094,215
1,069	106 822,461	3 267,185
	11,411 038,174 096,521	348 991,093 242,285
1,077	361	11
1,085	33	1
1,093	33	1
	1,091	33
1,101	365	11
1,109	106,865	3,209
	11,420 128,227	342 929,785
1,117	7 484,589	223,945
	56 019,072 498,923	1 676,136 283,605
1,125	15,127	451
1,133	101	3
1,141	1,275 183,065	37 751,109
1,165	1,809	53
	3 272,483	95,877
1,181	29,039	845
	843 263,523	24 537,955
1,189	25,689	745
	659 924,723	19 138,305
1,197	173	5
1,205	243	7
1,221	35	1
1,229	35	1
	1,227	35
1,237	1 294,047	36,793
	1 674,557 638,211	47,611 871,271
1,245	247	7
1,253	177	5
1,261	79,011	2,225

	6,242 738,123	175 799,475
1,277	6,611	185
	43 705,323	1 223,035
1,285	25,989	725
	675 428,123	18 842,025
1,309	117,115	3,237
1,317	421,877	11,625
1,333	87,077	2,385
1,341	13 860,727	378,505
1,349	15,977	435
1,357	892,609	24,231
1,365	37	1
1,373	37	1
	1,371	37
1,381	75,401 981,961	2,029 018,105
	5,685 458,883 646,969 405,523	152 991,986 551,752 403,905
1,397	3,177	85
1,413	45,371	1,207
1,429	189	5
	35,723	945
1,437	57,961	1,529
1,453	3,059 939,997	80 274,961
	9 363,232 785,240 360,011	245,636 563,921 515,117
1,461	9 409,325	246,169
1,469	115	3
1,477	27 193,889	707,589
1,493	2,357	61
	5 555,451	143,777
1,501	3,002 570,777	77 500,215
1,509	505	13
1,517	39	1
1,525	39	1
	1,523	39
1,533	509	13
1,541	1 185,165	30,191
1,549	676,923 333,555	17,199 418,961
	458,225 199,511 213,788 938,027	11,642 688,018 289,194 536,355
1,557	29,239	741
1,565	989	25
	978,123	24,725
1,573	119	3
1,581	835	21
1,589	6 330,805	158,817
1,597	100 646,511	2 518,525
	10,129 720,176 473,123	253 480,754 116,275
1,621	4 823,622 127,875	119,806 883,557
	23 267,330 432,525 342,852 015,627	577,903 134,597 288,688 851,375
1,629	1 703,027	42,195
1,637	5,543	137
	30 724,851	759,391
1,645	26,647	657
1,653	1,423	35
1,661	2 917,473	71,585
1,669	1,293 350,265	31 658,329

	1 672,754 907,975 570,227	40,945 308,201 607,185
1,677	41	1
1,685	41	1
	1,683	41
1,693	1 372,839	33,365
	1 884,686 919,923	45,804 773,235
1,709	5 391,115	130,409
	29 064,120 943,227	703,049 916,035
1,725	623	15
1,733	172,387	4,141
	29,717 277,771	713 854,567
1,741	85,889 757,675	2,058 457,913
	7,377 050,473 470,221 405,627	176 800,451 331,756 232,275
1,749	3,973	95
1,781	211	5
	44,523	1,055
1,789	548,890 789,515	12,977 193,281
	301,281 098,814 400,033 935,227	7,123 061,865 696,843 248,715
1,797	316,873	7,475
1,821	3,821 114,165	89 543,701
1,829	81 456,873	1 904,675
1,837	28 761,577	671,055
1,845	43	1
1,853	43	1
	1,851	43
1,869	25,723	595
1,877	1,603	37
	2 569,611	59,311
1,933	812 454,627	18 479,201
	660,082 520,933 709,131	15,013 512,355 713,027
1,941	10,523 512,585	238 862,491
1,957	929	21
1,965	133	3
1,981	9 856,153 532,405	221,444 665,221
1,989	223	5
1,997	9,161	205
	83 923,923	1 878,005

College of the City of New York.

ON THE RANK OF A MATRIX.

By W. H. Metzler.

1. It is a well-known theorem that: *If in a given matrix a certain m-rowed determinant is not zero, and all the* $(m + 1)$*-rowed determinants of which this m-rowed determinant is a first minor are zero, then all the* $(m + 1)$*-rowed determinants of the matrix are zero.*

The principal object of this paper is to give a proof of the following generalization* of the foregoing theorem and to give the corresponding generalizations for symmetric and skew-symmetric matrices:

If in a given matrix a certain m-rowed determinant is not zero, and all the $(m + h)$*-rowed determinants of which this m-rowed determinant is a minor are zero, then all the* $(m + h)$*-rowed determinants of the matrix are zero.*

Let

$$M \equiv \begin{vmatrix} (n \mid \underset{\alpha}{m}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}^{\dagger} \neq 0, \qquad \text{and} \qquad \begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\beta}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0,$$

for

$$\left.\begin{matrix} \beta \\ \gamma \end{matrix}\right\} = 1, 2, \cdots (n - m)_h.$$

On account of identical columns we obviously have

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\delta}{h + k}) \\ (n \mid \underset{\alpha}{m})(n \mid \underset{\alpha}{m} \mid \underset{i}{k})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} \equiv 0.$$

Expanding in terms of minors of order k and their complementaries we have

$$\sum_{j=1}^{m_k} \begin{vmatrix} (n \mid \underset{\alpha}{m} \mid \underset{j}{k}) \\ (n \mid \underset{\alpha}{m} \mid \underset{i}{k}) \end{vmatrix} \cdot \begin{vmatrix} (n \mid \underset{\alpha}{\bar{m}} \mid \underset{j}{k})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\delta}{h + k}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0,$$

for $i = 1, 2, \cdots m_k$, and $k = 0, 1, \cdots m$.

The determinant of the coefficients‡ in this set of equations for each

* This extension was proved by the writer in February, 1913, but before it was put in complete form for publication Cullis's book on Matrices and Determinoids appeared and contains the extension. While the theorem is no longer novel the method of proof here employed is of interest.

† Obviously no generality is lost in taking the m-rowed determinant coaxial.

‡ By coefficient here is meant the first factor in each term.

value of k is the kth compound of M and is therefore not zero. Giving k successively the values from 1 to m we have

$$\begin{vmatrix} (n \mid \underset{\alpha}{\overline{m}} \mid \underset{j}{k})(\overline{n} \mid \underset{\alpha}{m} \mid h \underset{\delta}{+} k) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0, \tag{1}$$

for all values of j, δ, γ, k.

For the values $1, 2, \cdots m$ of k we have the result that all minors of the matrix which have m columns and $(m - 1), (m - 2), \cdots 0$, rows in common with M vanish.

Therefore

$$\begin{vmatrix} (n \mid m \underset{\epsilon}{+} h) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0, \tag{2}$$

for $\epsilon = 1, 2, \cdots n_{m-h}$.

Lemma. If

$$\begin{vmatrix} (n \mid m \underset{\epsilon}{+} h) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0,$$

for all values of ϵ, γ, then

$$M' \equiv \begin{vmatrix} (n \mid m + \underset{\eta}{h} + k) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid h \underset{\delta}{+} k) \end{vmatrix} = 0. \tag{3}$$

That is if all minors of order $(m + h)$ formed from the m columns $(n \mid \underset{\alpha}{m})$ and h others vanish, then all minors of order $(m + h + k)$ formed from the same m columns and $(h + k)$ others will vanish.

For

$$M' = \sum_{i=1}^{(m+h)_k} \begin{vmatrix} (n \mid m + \underset{\eta}{h} + k \mid \underset{i}{k}) \\ (\overline{n} \mid \underset{\alpha}{m} \mid h \underset{\delta}{+} k \mid \underset{\rho}{k}) \end{vmatrix} \cdot \begin{vmatrix} (n \mid \overline{m + \underset{\eta}{h} + k} \mid \underset{i}{k}) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid \overline{h \underset{\delta}{+} k} \mid \underset{\rho}{k}) \end{vmatrix} = 0,$$

since the minors of order $(m + h)$ in every term are zero by hypothesis.

Since by (3) $M' = 0$, it is obvious that

$$\begin{vmatrix} (n \mid \underset{\alpha}{m} \mid \underset{j}{k})(n \mid m \underset{\epsilon}{+} h) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid h \underset{\delta}{+} k) \end{vmatrix} = 0,$$

and we have

$$\sum_{i=1}^{m_k} \begin{vmatrix} (n\mid \underset{\alpha}{m}\mid \underset{j}{k}) \\ (n\mid \underset{\alpha}{m}\mid \underset{i}{k})\end{vmatrix} \cdot \begin{vmatrix} (n\mid \underset{\epsilon}{m+h}) \\ (n\mid \underset{\alpha}{\overline{m}}\mid \underset{i}{k})(\overline{n}\mid \underset{\alpha}{m}\mid \underset{\delta}{h+k})\end{vmatrix} = 0$$

for $k = 1, 2, \cdots m$, and $j = 1, 2, \cdots m_k$.

The determinant of the coefficients in this set of equations is the kth compound of M and is therefore not zero.

It follows that

$$\begin{vmatrix} (n\mid \underset{\epsilon}{m+h}) \\ (n\mid \underset{\alpha}{\overline{m}}\mid \underset{i}{k})(\overline{n}\mid \underset{\alpha}{m}\mid \underset{\delta}{h+k})\end{vmatrix} = 0 \tag{4}$$

for all values of ϵ, i, δ, k.

Giving k successively the values $1, 2, \cdots m$ we have our theorem proved, or stated symbolically it is:

If

$$\begin{vmatrix} (n\mid \underset{\alpha}{m}) \\ (n\mid \underset{\alpha}{m})\end{vmatrix} \neq 0, \qquad \text{and} \qquad \begin{vmatrix} (n\mid \underset{\alpha}{m})(\overline{n}\mid \underset{\alpha}{m}\mid \underset{\beta}{h}) \\ (n\mid \underset{\alpha}{m})(\overline{n}\mid \underset{\alpha}{m}\mid \underset{\gamma}{h})\end{vmatrix} = 0,$$

for

$$\left.\begin{matrix}\beta\\ \gamma\end{matrix}\right\} = 1, 2, \cdots (n-m)_h,$$

then

$$\begin{vmatrix} (n\mid \underset{\alpha}{\overline{m}}\mid \underset{i}{g})(\overline{n}\mid \underset{\alpha}{m}\mid \underset{\delta}{h+g}) \\ (n\mid \underset{\alpha}{\overline{m}}\mid \underset{j}{k})(\overline{n}\mid \underset{\alpha}{m}\mid \underset{\epsilon}{h+k})\end{vmatrix} = 0, \tag{5}$$

for

$$\left.\begin{matrix}g\\ k\end{matrix}\right\} = 0, 1, \cdots m, \qquad \left.\begin{matrix}i\\ j\end{matrix}\right\} = 1, 2, \cdots m_k, \qquad \left.\begin{matrix}\delta\\ \epsilon\end{matrix}\right\} = 1, 2, \cdots (n-m)_{h+k}.$$

2. In a symmetric determinant, if all coaxial minors of order $m + h$ and $m + h + 1$, which contain the coaxial minor M of order m, are zero; then all minors of order $m + h$, which contain M, will be zero.

That is if

$$M \equiv \begin{vmatrix} (n\mid \underset{\alpha}{m}) \\ (n\mid \underset{\alpha}{m})\end{vmatrix},$$

and if

$$\begin{vmatrix} (n\mid \underset{\alpha}{m})(\overline{n}\mid \underset{\alpha}{m}\mid \underset{\beta}{h}) \\ (n\mid \underset{\alpha}{m})(\overline{n}\mid \underset{\alpha}{m}\mid \underset{\beta}{h})\end{vmatrix} = 0$$

and

$$D \equiv \begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h + 1}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h + 1}) \end{vmatrix} = 0$$

for all values of β and γ; then

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix} = 0 \tag{6}$$

for all values of i and j.

For

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \end{vmatrix} \cdot \begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix} - \begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix}^2$$

is a coaxial minor of order two of the reciprocal of D and is therefore zero, and since each factor in the first term is zero it follows that

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix} = 0.$$

If (6) is true and $M \neq 0$, it follows from (5) that every minor of order $m + h$ of a symmetric determinant is zero.

3. In a skew-symmetric determinant if the coaxial minor

$$M \equiv \begin{vmatrix} (n \mid \underset{\alpha}{m}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}$$

is of even order, and if the coaxial minors of even order

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\beta}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\beta}{h}) \end{vmatrix} = 0,$$

for all values of β; then

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix} = 0 \tag{7}$$

for all values of i and j.

For since $m + h + 1$ is odd, every coaxial minor of that order is zero. It follows then as in article 2 that every minor of the $(m + h)$th order which contains M is zero.

If (7) is true, and if $M \neq 0$, it follows from (5) that every minor of order $m + h$ of a skew-symmetric determinant is zero.

For other conditions determining the rank of a symmetric and a skew-symmetric determinant see the author's paper* in the American Journal of Mathematics for 1894.

Syracuse University,
November, 1913.

* Metzler, Compound Determinants, Amer. Jour. Math., vol. 16, pp. 131–150.

ON THE EXPRESSION OF CERTAIN MINORS OF THE LTH COMPOUND OF A DETERMINANT AS A FUNCTION OF THE ELEMENTS OF A SINGLE LINE OF THE MTH COMPOUND.

By W. H. Metzler.

In the Messenger of Mathematics for December, 1905, Dr. Thos. Muir called attention to a relation between two determinants of orders n and $(n - 1)$, which arose in connection with a problem of elimination. In this paper a slightly more general relation is considered from a somewhat different angle, adding sufficient novelty for further interest.

If

$$\Delta \equiv \left(\begin{vmatrix} (l_1) \\ (n \mid \underset{\alpha}{m} \mid \underset{1}{l}) \end{vmatrix} \begin{vmatrix} (l_2) \\ (n \mid \underset{\alpha}{m} \mid \underset{2}{l}) \end{vmatrix} \cdots \begin{vmatrix} (l_\lambda) \\ (n \mid \underset{\alpha}{m} \mid \underset{\lambda}{l}) \end{vmatrix} \right),$$

and

$$\Delta' \equiv \left(\begin{vmatrix} (n \mid \underset{\beta}{\overline{m}} \mid \underset{1}{l}) \\ (n \mid \underset{\alpha}{\overline{m}} \mid \underset{1}{l}) \end{vmatrix} \begin{vmatrix} (n \mid \underset{\beta}{\overline{m}} \mid \underset{2}{l}) \\ (n \mid \underset{\alpha}{m} \mid \underset{2}{l}) \end{vmatrix} \cdots \begin{vmatrix} (n \mid \underset{\beta}{\overline{m}} \mid \underset{\lambda}{l}) \\ (n \mid \underset{\alpha}{m} \mid \underset{\lambda}{l}) \end{vmatrix} \right),$$

where $(l_1), (l_2), \cdots, (l_\lambda)$ are λ sets of l numbers each taken from the numbers $1, 2, 3 \cdots, n$; and where no two sets have any numbers in common; then the product

$$\Delta \cdot \Delta' = \Delta'' \equiv \left(\begin{vmatrix} (l_1)(n \mid \underset{\beta}{\overline{m}} \mid \underset{1}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \begin{vmatrix} (l_2)(n \mid \underset{\beta}{\overline{m}} \mid \underset{2}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \cdots \begin{vmatrix} (l_\lambda)(n \mid \underset{\beta}{\overline{m}} \mid \underset{\lambda}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \right). \quad (1)$$

It may be observed that while Δ is a minor of the lth compound of a given determinant, and Δ' is a minor of the $(m - l)$th compound of this same determinant, Δ'' is not a minor of the mth compound of this determinant, but a function of the elements of a single column of the mth compound.

It should also be observed that the row numbers of Δ'' are the row numbers of Δ combined with the corresponding row numbers of Δ'.

Suppose now that $m = k \cdot l + h$, where $h < l$ and therefore k is the greatest integer in m/l.

Let

$$(l_1)(l_2)(l_3) \cdots (l_k)(h) = (n \mid \underset{\beta}{m}),$$

then obviously

$$\Delta'' = \Delta''' \begin{vmatrix} (n \mid \underset{\beta}{m}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}^{k}, \tag{2}$$

where

$$\Delta''' \equiv \left(\begin{vmatrix} (l_{k+1})(n \mid \underset{\beta}{\overline{m}} \mid \underset{k+1}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \cdots \begin{vmatrix} (l_\lambda)(n \mid \underset{\beta}{\overline{m}} \mid \underset{\lambda}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \right).$$

Again, since

$$\Delta' = \begin{vmatrix} (n \mid \underset{\beta}{m}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}^{(m-1)_l} \equiv M^{(m-1)_l},$$

say, we have

$$\Delta \cdot M^{(m-1)_l - k} = \Delta'''. \tag{3}$$

This may be looked upon as a relation giving the value of Δ in terms of Δ''', or giving the value of Δ''' in terms of Δ.

If in (3) $l = m - 1$, then $k = 1$, and $h = 1$, and we have

$$\Delta = \Delta''', \tag{4}$$

a result given by Muir.*

If

$$(l_1) \equiv (\alpha_1\alpha_2 \cdots \alpha_{m-1}) \equiv (n \mid \underset{\beta}{m} \mid \underset{1}{l}),$$
$$(l_2) \equiv (\alpha_2\alpha_3 \cdots \alpha_m),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(l_\lambda) \equiv (\alpha_m\alpha_{m+1} \cdots \alpha_{2m-1}),$$

then

$$\Delta = \Delta''' = \begin{vmatrix} (\alpha_1\alpha_2 \cdots \alpha_m) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \cdot \begin{vmatrix} (\alpha_2\alpha_3 \cdots \alpha_{m+1}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \cdots \begin{vmatrix} (\alpha_m\alpha_{m+1} \cdots \alpha_{2m-1}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}^{*}. \tag{5}$$

As an example of (3) let $m = 4$, $l = 2$, $(n \mid \underset{\beta}{m}) \equiv (9\tau\sigma\rho)$, and $(n \mid \underset{\alpha}{m}) \equiv (1234)$ and we have

$$\left(\begin{vmatrix} 12 \\ 12 \end{vmatrix} \begin{vmatrix} 34 \\ 13 \end{vmatrix} \begin{vmatrix} 56 \\ 14 \end{vmatrix} \begin{vmatrix} 78 \\ 23 \end{vmatrix} \begin{vmatrix} 9\tau \\ 24 \end{vmatrix} \begin{vmatrix} \sigma\rho \\ 34 \end{vmatrix} \right) \times \begin{vmatrix} 9\tau\sigma\rho \\ 1234 \end{vmatrix} = \left(\begin{vmatrix} 12\tau\rho \\ 1234 \end{vmatrix} \begin{vmatrix} 34\tau\sigma \\ 1234 \end{vmatrix} \begin{vmatrix} 569\rho \\ 1234 \end{vmatrix} \begin{vmatrix} 789\sigma \\ 1234 \end{vmatrix} \right).$$

Here $k = 2$, and the row numbers of Δ''' are formed by adding to the row numbers 12, 34, 56, 78, of Δ, the row numbers $\tau\rho$, $\tau\sigma$, 9ρ, 9σ, which are the combinations of $9\tau\sigma\rho$ taken two at a time except those two which appear in Δ, viz. 9τ and $\sigma\rho$.

If in this example we had taken $(n \mid \underset{\beta}{m}) \equiv (1234)$ we would have

$$\left(\begin{vmatrix} 12 \\ 12 \end{vmatrix} \begin{vmatrix} 34 \\ 13 \end{vmatrix} \begin{vmatrix} 56 \\ 14 \end{vmatrix} \begin{vmatrix} 78 \\ 23 \end{vmatrix} \begin{vmatrix} 9\tau \\ 24 \end{vmatrix} \begin{vmatrix} \sigma\rho \\ 34 \end{vmatrix} \right) \begin{vmatrix} 1234 \\ 1234 \end{vmatrix} = \left(\begin{vmatrix} 1356 \\ 1234 \end{vmatrix} \begin{vmatrix} 1478 \\ 1234 \end{vmatrix} \begin{vmatrix} 239\tau \\ 1234 \end{vmatrix} \begin{vmatrix} 24\sigma\rho \\ 1234 \end{vmatrix} \right).$$

* Ibid. Cf. author's paper on Compound Determinants, Amer. Jour. Math., vol. 20, No. 3.

If each element in the first column of Δ is the sum of two terms we would have

$$\left(\begin{vmatrix}12\\12\end{vmatrix} + \begin{vmatrix}pq\\12\end{vmatrix}\begin{vmatrix}34\\13\end{vmatrix}\begin{vmatrix}56\\14\end{vmatrix}\begin{vmatrix}78\\23\end{vmatrix}\begin{vmatrix}9\tau\\24\end{vmatrix}\begin{vmatrix}\sigma\rho\\34\end{vmatrix}\right)\times\begin{vmatrix}9\tau\sigma\rho\\1234\end{vmatrix}$$

$$= \left(\begin{vmatrix}12\tau\rho\\1234\end{vmatrix} + \begin{vmatrix}pq\tau\rho\\1234\end{vmatrix}\begin{vmatrix}34\tau\sigma\\1234\end{vmatrix}\begin{vmatrix}569\rho\\1234\end{vmatrix}\begin{vmatrix}789\sigma\\1234\end{vmatrix}\right).$$

In general, if

$$S_k = \sum_{i=1}^{h_k}\begin{vmatrix}(l_k{}^i)\\ (n \mid \underset{\alpha}{m} \mid \underset{k}{l})\end{vmatrix},$$

and

$$\Delta \equiv (S_1 S_2 \cdots S_\lambda),$$

then

$$\Delta \cdot \Delta' = \Delta'' = (S_1{}'S_2{}' \cdots S_\lambda{}'), \tag{6}$$

where

$$S_k{}' = \sum_{i=1}^{h_k}\begin{vmatrix}(l_k{}^i)(n \mid \underset{\beta}{\overline{m}} \mid \underset{k}{l})\\ (n \mid \underset{\alpha}{m})\end{vmatrix}.$$

A few further examples will illustrate with what ease the value of such determinants may be written down.

Taking $l = m - 1$, we have

$$\begin{vmatrix}
\begin{vmatrix}123\\123\end{vmatrix}\begin{vmatrix}126\\123\end{vmatrix} - \begin{vmatrix}135\\123\end{vmatrix} + \begin{vmatrix}234\\123\end{vmatrix}\begin{vmatrix}156\\123\end{vmatrix} - \begin{vmatrix}246\\123\end{vmatrix} + \begin{vmatrix}345\\123\end{vmatrix}\begin{vmatrix}456\\123\end{vmatrix}\\
\begin{vmatrix}123\\124\end{vmatrix}\begin{vmatrix}126\\124\end{vmatrix} - \begin{vmatrix}135\\124\end{vmatrix} + \begin{vmatrix}234\\124\end{vmatrix}\begin{vmatrix}156\\124\end{vmatrix} - \begin{vmatrix}246\\124\end{vmatrix} + \begin{vmatrix}345\\124\end{vmatrix}\begin{vmatrix}456\\124\end{vmatrix}\\
\begin{vmatrix}123\\134\end{vmatrix}\begin{vmatrix}126\\134\end{vmatrix} - \begin{vmatrix}135\\134\end{vmatrix} + \begin{vmatrix}234\\134\end{vmatrix}\begin{vmatrix}156\\134\end{vmatrix} - \begin{vmatrix}246\\134\end{vmatrix} + \begin{vmatrix}345\\134\end{vmatrix}\begin{vmatrix}456\\134\end{vmatrix}\\
\begin{vmatrix}123\\234\end{vmatrix}\begin{vmatrix}126\\234\end{vmatrix} - \begin{vmatrix}135\\234\end{vmatrix} + \begin{vmatrix}234\\234\end{vmatrix}\begin{vmatrix}156\\234\end{vmatrix} - \begin{vmatrix}246\\234\end{vmatrix} + \begin{vmatrix}345\\234\end{vmatrix}\begin{vmatrix}456\\234\end{vmatrix}
\end{vmatrix}$$

$$= \begin{vmatrix}
\begin{vmatrix}1234\\1234\end{vmatrix}\begin{vmatrix}1345\\1234\end{vmatrix} - \begin{vmatrix}1246\\1234\end{vmatrix}\begin{vmatrix}1456\\1234\end{vmatrix}\\
\begin{vmatrix}1235\\1234\end{vmatrix}\begin{vmatrix}2345\\1234\end{vmatrix} - \begin{vmatrix}1256\\1234\end{vmatrix}\begin{vmatrix}2456\\1234\end{vmatrix}\\
\begin{vmatrix}1236\\1234\end{vmatrix}\begin{vmatrix}2346\\1234\end{vmatrix} - \begin{vmatrix}1356\\1234\end{vmatrix}\begin{vmatrix}3456\\1234\end{vmatrix}
\end{vmatrix}^{*}.$$

* Muir, ibid.

Similarly the determinant whose elements in the first row are

$$\begin{vmatrix}123\\123\end{vmatrix},\ \begin{vmatrix}145\\123\end{vmatrix}+\begin{vmatrix}246\\123\end{vmatrix}+\begin{vmatrix}356\\123\end{vmatrix},\ \begin{vmatrix}345\\123\end{vmatrix}+\begin{vmatrix}146\\123\end{vmatrix}+\begin{vmatrix}256\\123\end{vmatrix},\ \begin{vmatrix}456\\123\end{vmatrix}$$

$$=\begin{vmatrix}\begin{vmatrix}1234\\1234\end{vmatrix} & \begin{vmatrix}3456\\1234\end{vmatrix} & \begin{vmatrix}2456\\1234\end{vmatrix}\\[2ex] \begin{vmatrix}1235\\1234\end{vmatrix} & -\begin{vmatrix}2456\\1234\end{vmatrix} & -\begin{vmatrix}1456\\1234\end{vmatrix}\\[2ex] \begin{vmatrix}1236\\1234\end{vmatrix} & \begin{vmatrix}1456\\1234\end{vmatrix} & \begin{vmatrix}3456\\1234\end{vmatrix}\end{vmatrix};$$

and the determinant whose elements in the first row are

$$\begin{vmatrix}123\\123\end{vmatrix},\ \begin{vmatrix}145\\123\end{vmatrix}+\begin{vmatrix}146\\123\end{vmatrix}+\begin{vmatrix}156\\123\end{vmatrix},\ \begin{vmatrix}245\\123\end{vmatrix}+\begin{vmatrix}246\\123\end{vmatrix}+\begin{vmatrix}256\\123\end{vmatrix},\ \begin{vmatrix}456\\123\end{vmatrix}$$

$$=\begin{vmatrix}\begin{vmatrix}1234\\1234\end{vmatrix} & \begin{vmatrix}1456\\1234\end{vmatrix} & \begin{vmatrix}2456\\1234\end{vmatrix}\\[2ex] \begin{vmatrix}1235\\1234\end{vmatrix} & -\begin{vmatrix}1456\\1234\end{vmatrix} & -\begin{vmatrix}2456\\1234\end{vmatrix}\\[2ex] \begin{vmatrix}1236\\1234\end{vmatrix} & \begin{vmatrix}1456\\1234\end{vmatrix} & \begin{vmatrix}2456\\1234\end{vmatrix}\end{vmatrix}=0.$$

SYRACUSE UNIVERSITY,
March 1, 1914.

IMPLICIT FUNCTIONS AT A BOUNDARY POINT.

By L. S. Dederick.

1. Introduction. The ordinary theorems* which state the existence of implicit functions of real variables defined as the solution of one or more equations, demand that the initial point or known solution shall be an *interior* point of the region of definition of the functions in the given equations. In other words, the existence, continuity, etc., of the functions occurring in the equations to be solved, must be given in a *complete* neighborhood of the initial point.

Moreover, it is easy to see that these theorems do not hold at a boundary point. For example, the function $y + x^2$ defined in the region $y \geqq 0$, vanishes at the origin but at no other point of the region, although it satisfies all the conditions which would be sufficient at an interior point to insure the existence of a continuous solution of the equation $y + x^2 = 0$ in the neighborhood. In fact it would be obvious without an example that the boundary might be so drawn as to cut off the solution.

It is scarcely obvious, however, that the ordinary conditions may be fulfilled at a boundary point, and yet the solution may not be single-valued in the neighborhood. This is illustrated by the following example. Let $F(x, y)$ be defined in the region $x \leqq \sqrt{|y|}$, being equal to y when $x \leqq 0$, equal to $y - 2x^2$ when $x \geqq 0$ and $y \geqq x^2$, and equal to $y + 2x^2$ when $x \geqq 0$ and $y \leqq -x^2$. This function is easily seen to be continuous, together with its first order derivatives, within and on the boundary of the region of definition, and $\partial F/\partial y \neq 0$ at the origin. Yet the solution of $F(x, y) = 0$ is obviously two-valued when $x > 0$, since it consists of $y = 2x^2$ and $y = -2x^2$.

These examples show that in the ordinary theorems the restriction of the initial solution to an interior point is an essential one. The purpose of this article is to derive theorems for the case where the known solution occurs at a boundary point of the region of definition of the given functions, still retaining the condition that the Jacobian of these functions shall be different from zero at the initial point. The method of treatment agrees with that of *Kowalewski*† in that it proceeds without induction and reverses

* Cf. G. A. Bliss, Princeton Colloquium Lectures, § 1.

† Leipziger Berichte, vol. 60 (1908), pp. 10–19. This paper has evidently been overlooked by Bliss. For his methods for establishing existence and uniqueness are practically the same as those given by Kowalewski in this article, although the latter is not referred to either in the Colloquium Lectures or in Professor Bliss's earlier presentation of these methods in his article in the Bulletin of the American Mathematical Society, vol. 18 (1912), p. 175.

the more usual order by treating first the inversion of a transformation and deriving from this the solution of a system of equations.

2. Preliminary Definitions. The notation of Bolza,* by which a function is said to be *of class* C' if it is one-valued and continuous and has its derivative (or all its partial derivatives) of the first order continuous, will be extended to transformations and systems of equations if all the functions appearing explicitly in them are of class C'. A transformation or system of equations of class C' will be called *regular* at any point or in any region where its Jacobian is different from zero.

By a *region*, unless the contrary is stated, will always be meant a complete region, i. e. a continuum together with all its boundary points. Moreover, at any boundary point of a region at which a given partial derivative can not be formed, that derivative of any function will be called continuous if its value in the interior of the region approaches a unique boundary value, and this will be called the value of the derivative at the boundary point.

A region will be called *uniformly connected* if there exists a constant G such that any two points of the region can be connected by a broken line lying wholly in the region and having a length not greater than G times the distance in a straight line between the points. An example of a region which is not uniformly connected is furnished by the portion of a plane external to a cusp.

A function of n variables, $f(x_1, \cdots x_n)$, will be called *uniformly differentiable* in a region if there exist n variables $\alpha_1, \cdots \alpha_n$, such that

$$f(a_1 + h_1, \cdots a_n + h_n) - f(a_1, \cdots a_n) = \sum_{i=1}^{n} h_i[f_{x_i}(a_1, \cdots a_n) + \alpha_i],$$

and such that the α's approach zero with the h's uniformly for all values of $a_1, \cdots a_n$ in the region.

By the *vicinity*† of a boundary point of a region will be meant that part of a complete neighborhood of the point which belongs to the region, or if no region is mentioned, then merely some part or perhaps all of a complete neighborhood.

3. Lemma on Uniformly Differentiable Functions. *A function of class C' in a uniformly connected region including its boundary, is uniformly differentiable in this region.*

The proof will be given for two variables, but is exactly the same for any number.

Let $f(x, y)$ be the function and (a, b) be any point of the region, either

* Calculus of Variations, pp. 6, 7.

† Cf. Pierpont, Theory of functions of real variables, vol. 1, p. 155.

in the interior or on the boundary. Then it is to be shown that α and β exist such that

$$(1) \quad \Delta f = f(a+h, b+k) - f(a, b) = h(f_x(a, b) + \alpha) + k(f_y(a, b) + \beta),$$

and such that α and β each approach zero when h and k do, and that this approach is uniform for all values of a and b in the region; i. e. to any positive ϵ there corresponds a δ independent of (a, b) such that $|\alpha| < \epsilon$ and $|\beta| < \epsilon$ whenever $\sqrt{h^2 + k^2} < \delta$.

Let the vertices of the broken line joining (a, b) to $(a + h, b + k)$ which is demanded by the fact that the region is uniformly connected, be $(x_0 = a, y_0 = b)$, (x_1, y_1), (x_2, y_2), $\cdots$ $(x_n = a + h, y_n = b + k)$. Then by applying Taylor's theorem to each of the segments and adding, we get

$$(2) \qquad \Delta f = \sum_{i=1}^{n} [(x_i - x_{i-1}) f_x(x_i', y_i') + (y_i - y_{i-1}) f_y(x_i', y_i')],$$

where (x_i', y_i') is some point on the segment joining (x_{i-1}, y_{i-1}) to (x_i, y_i). If we write $f_x(x_i', y_i') = f_x(a, b) + \alpha_i$ and $f_y(x_i', y_i') = f_y(a, b) + \beta_i$ and observe that $\sum_{i=1}^{n} (x_i - x_{i-1}) = h$ and $\sum_{i=1}^{n} (y_i - y_{i-1}) = k$, we get by comparing (1) and (2) that

$$(3) \qquad h\alpha + k\beta = \sum_{i=1}^{n} [\alpha_i (x_i - x_{i-1}) + \beta_i (y_i - y_{i-1})].$$

Since f_x and f_y are continuous in the complete region they are uniformly continuous. This enables us to choose δ independent of (a, b) such that if $\sqrt{(x-a)^2 + (y-b)^2} < G\delta$, then $|f_x(x, y) - f_x(a, b)| < \epsilon/2G$ and $|f_y(x, y) - f_y(a, b)| < \epsilon/2G$. If $\sqrt{h^2 + k^2} < \delta$, then $\sqrt{(x_i' - a)^2 + (y_i' - b)^2} < G\delta$ $(i = 1, \cdots n)$, and hence $|\alpha_i| < \epsilon/2G$ and $|\beta_i| < \epsilon/2G$. Moreover $\sum_{i=1}^{n} |x_i - x_{i-1}| \leqq G\sqrt{h^2 + k^2}$ and $\sum_{i=1}^{n} |y_i - y_{i-1}| \leqq G\sqrt{h^2 + k^2}$. Applying these relations to (3), we get

$$|h\alpha + k\beta| < \epsilon \sqrt{h^2 + k^2}.$$

If we choose α and β proportional to h and k we may write $\alpha = \rho h$, $\beta = \rho k$, $h\alpha + k\beta = \rho(h^2 + k^2)$. Therefore

$$|\rho| (h^2 + k^2) < \epsilon \sqrt{h^2 + k^2},$$

$$|\rho| < \frac{\epsilon}{\sqrt{h^2 + k^2}},$$

$$|\alpha| = |\rho h| < \frac{|h|}{\sqrt{h^2 + k^2}} \epsilon \leqq \epsilon.$$

Similarly $|\beta| < \epsilon$, and the lemma is proved.

An example of a function of class C' but not uniformly differentiable in a region is given by the function $F(x, y)$ defined in § 1.

Incidentally we have derived by this lemma a sufficient condition that a function of two or more variables be totally differentiable at a boundary point of a region in which it is known to be of class C'. This property, however, can be established under a much less stringent condition, which we may state by saying that the boundary point must be 'uniformly accessible,' i. e. the broken line joining it to any point in the vicinity must be less than G times its distance from the point, where G is a constant.

4. The Inverse of a Transformation. THEOREM 1. *If a transformation*

$$T: \qquad y_i = f_i(x_1, \cdots x_n), \qquad i = 1, \cdots n$$

is regular in a uniformly connected region S, of which $a = (a_1, \cdots a_n)$ is a boundary point carried by T into a point $b = (b_1, \cdots b_n)$, then there exists a vicinity of a, $V(a)$, in which T possesses an inverse

$$T^{-1}: \qquad x_i = \varphi_i(y_1, \cdots y_n), \qquad i = 1, \cdots n$$

which is regular.

To prove this we must show first that if $V(a)$ is suitably restricted, T^{-1} is one-valued, i. e. no two points of $V(a)$ are carried by T into the same point. If two points $x = (x_1, \cdots x_n)$ and $x' = (x_1', \cdots x_n')$ are transformed by T into the same point, the increment Δf_i of each f_i in passing from one point to the other must be zero. Now by the Lemma,

$$(1) \qquad \Delta f_i = \sum_{j=1}^{n} (x_j' - x_j)(f_{ij}(x) + \alpha_{ij}), \qquad i = 1, \cdots n$$

where $f_{ij} = \partial f_i/\partial x_j$ and each α_{ij} approaches zero as x' approaches x, the approach being uniform for all points x. Hence by taking $V(a)$ sufficiently small, the maximum value of $x_j' - x_j$ will be restricted so that each α_{ij} is arbitrarily small. This, together with the continuity of the functions f_{ij} and the continuity of a determinant as a function of its elements,* shows that in a sufficiently small vicinity of a the determinant of the coefficients of $x_j' - x_j$ in (1) will remain different from zero. Hence setting $\Delta f_1 = \Delta f_2 = \cdots = \Delta f_n = 0$ makes $x_j' = x_j$ $(j = 1, \cdots n)$; and two points having the same image must be identical.

The proofs that the functions φ_i are continuous and have continuous

* Cf. Bliss, op. cit., p. 9, where it is stated that essentially this same determinant will remain different from zero in any region in which the Jacobian remains different from zero, the fact being apparently overlooked that the arguments are in general different in the elements of different rows. This is obviously an oversight as it would imply directly that a transformation with a non-vanishing Jacobian would have a one-valued inverse for any convex region as a whole.

partial derivatives, are identical with those given by Kowalewski* for an interior point, so that it is not worth while to reproduce them here. When these are supplied, and the Jacobian of T^{-1} shown in the ordinary way to be different from zero, the proof is complete.

This theorem states properties which are practically the same at a boundary point as at an interior point. Under these circumstances the theorem must necessarily lack one feature of the theorems for an interior point, namely a proof of existence; for this is essentially different at a boundary point. This topic is discussed in the two following theorems.

THEOREM 2. *If the transformation T is subject to the conditions of Theorem 1, the image of the boundary of S in the neighborhood of a divides the neighborhood of b into two parts, of which it forms the common boundary, such that at every point of one part the inverse T^{-1} exists, but at no point of the other.*

Let s denote the boundary of S, and s' its image; and let b' and b'' be any two points in the neighborhood of b but not on s'. Then if T^{-1} exists at b' and not at b'', we must show that any curve joining b' and b'' and remaining in the neighborhood of b, must meet s'. Since b' is not on s' it is the image of an interior point of S, and is therefore surrounded by a complete neighborhood in which T^{-1} exists. Since T^{-1} does not exist at b'', there must be a boundary point d on the curve such that T^{-1} exists at all points of the curve between b' and d, but fails to exist at some point of the curve in any arbitrarily small neighborhood of d. As a point moves along the curve from b' to d, it will have an image in S, which from the continuity of T^{-1} will move along some curve from a' to some limiting point c, which from the continuity of T must have the image d. The point c can not be an interior point of S, because in that case T^{-1} would exist in a complete neighborhood of d. It is therefore a point of s and hence d is on s'. This completes the proof.

It is easily shown in like manner that the image of S in the neighborhood of a forms a connected region.

THEOREM 3. *If a is a non-singular point of s, the boundary locus of S, Theorems 1 and 2 are applicable, and b, the image of a, is a non-singular point of s', the image of s, provided T is restricted to a sufficiently small neighborhood of a.*

To prove the first statement we merely need to observe that in the neighborhood of a the region S must obviously be uniformly connected.

To prove the second statement we may assume without loss of generality that the boundary s in the neighborhood of a has the form $x_1 = \omega(x_2, \cdots x_n)$,

* Loc. cit., §§ 3, 4. These proofs, however, are very simple and may easily be supplied by the reader.

where ω is of class C'. Then its image s' will be

$$\left.\begin{aligned} y_i &= f_i(\omega(x_2, \cdots x_n), x_2, \cdots x_n) \\ &= \psi_i(x_2, \cdots x_n) \end{aligned}\right\} \qquad i = 1, \cdots n,$$

being written in parametric form. It is readily seen that the functions ψ_i are of class C', and that consequently a necessary condition for a singularity of s' (for sufficiently restricted values of the parameters) is that the matrix of the partial derivatives of the functions ψ_i shall be of rank lower than $n - 1$. But since

$$\frac{\partial \psi_i}{\partial x_j} = \frac{\partial f_i}{\partial x_1}\frac{\partial \omega}{\partial x_j} + \frac{\partial f_i}{\partial x_j} = 0 \qquad \begin{aligned} i &= 1, \cdots n, \\ j &= 2, \cdots n, \end{aligned}$$

this would make $J = 0$ because this matrix can be obtained by an elementary transformation from that formed by striking out the first column of J. Therefore b is a non-singular point of s', the new boundary.

These theorems are more than sufficient to justify certain statements on this subject which were made without proof in a recent article* by the author on transformations with vanishing Jacobian.

It is easy to derive various other details of the properties of a regular transformation in a uniformly connected vicinity of a boundary point. A number of them may be roughly summed up in the statement that at such a point, as at an interior point, a regular transformation is approximately projective.

5. The Solution of a System of Equations. The problem of solving a system of equations is easily expressible in terms of finding the inverse of a transformation, whether it be at an interior point or at a boundary point. Let it be required to solve for the x's the equations

$$(1) \qquad F_i(x_1, \cdots x_n; u_1, \cdots u_m) = 0, \qquad i = 1, \cdots n.$$

This will be accomplished if we can find the inverse of the transformation

$$U: \quad \begin{aligned} y_i &= F_i(x_1, \cdots x_n; u_1, \cdots u_m), && i = 1, \cdots n, \\ v_i &= u_i, && i = 1, \cdots m, \end{aligned}$$

in the form

$$U^{-1}: \quad \begin{aligned} x_i &= \Phi_i(y_1, \cdots y_n; v_1, \cdots v_m), && i = 1, \cdots n, \\ u_i &= v_i, && i = 1, \cdots m. \end{aligned}$$

For we can then set $y_1 = y_2 = \cdots = y_n = 0$ and get the solution of (1) in the form

$$x_i = \Phi_i(0, \cdots 0; u_1, \cdots u_m), \qquad i = 1, \cdots n.$$

* Am. Math. Soc. Trans., vol. 14, pp. 143–148, the second footnote and the last.

Accordingly, if we merely observe that the Jacobian of U has the same value as that of (1), we may from Theorem 1 immediately derive

THEOREM 4. *If the system of equations* (1) *is regular in a uniformly connected region* S, *of which* $(a, c) = (a_1, \cdots a_n; c_1, \cdots c_m)$ *is a boundary point, and if* (a, c) *is a particular solution of* (1), *then in whatever vicinity of* $c = (c_1, \cdots c_m)$ *a solution of* (1) *exists, it is of the form*

$$x_i = f_i(u_1, \cdots u_m), \qquad i = 1, \cdots n, \tag{2}$$

where the functions f_i *are all of class* C', *provided the vicinity forms part of a sufficiently small neighborhood.*

It now remains to determine for what values of $u_1, \cdots u_m$ the functions f_i are defined. The following theorem answers this question in general for the case that (a, c) is a non-singular point of the boundary.

THEOREM 5. *If the system of equations* (1) *satisfies the conditions of Theorem* 4, *where the boundary of* S *is defined by the equation*

$$\Omega(x_1, \cdots x_n; u_1, \cdots u_m) = 0 \tag{3}$$

in the neighborhood of (a, c), Ω *being of class* C', *and if at this point the matrix*

$$\begin{Vmatrix} \frac{\partial F_1}{\partial x_1} \cdots \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial u_1} \cdots \frac{\partial F_1}{\partial u_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{\partial F_n}{\partial x_1} \cdots \frac{\partial F_n}{\partial x_n} & \frac{\partial F_n}{\partial u_1} \cdots \frac{\partial F_n}{\partial u_m} \\ \frac{\partial \Omega}{\partial x_1} \cdots \frac{\partial \Omega}{\partial x_n} & \frac{\partial \Omega}{\partial u_1} \cdots \frac{\partial \Omega}{\partial u_m} \end{Vmatrix}$$

is of rank $n + 1$, *then the solution* (2) *exists in a vicinity of* $c = (c_1, \cdots c_m)$ *of which the boundary may be written* $\theta(u_1, \cdots u_m) = 0$, *where* θ *is a function of class* C', *which at* c *is equal to zero but has some non-vanishing partial derivative.*

Since J, the Jacobian of the functions F_i with regard to the x's, is not zero, the matrix will contain a non-vanishing $(n + 1)$-rowed determinant of which J is a minor.* Let us call this K and assume as a matter of notation that the additional column in K is that containing $\partial F_1/\partial u_1, \cdots$. Some element of the last row in K must be different from zero, and we shall assume, also as matter of notation, that this is $\partial\Omega/\partial x_1$. (The case that the only non-vanishing element in the last row is $\partial\Omega/\partial u_1$ is not essentially different; but it will be considered separately.)

With these assumptions we may solve equation (3) for x_1, getting the

* Bôcher, Introduction to Higher Algebra, § 19, Theorem 1.

boundary of S in the form $x_1 = \omega(x_2, \cdots x_n; u_1, \cdots u_m)$, in a suitably restricted neighborhood (as always). If we apply to this the transformation U, we get the boundary of S', the image of S, written in parametric form,

$$\begin{aligned} y_i &= F_i(\omega(x_2, \cdots x_n; u_1, \cdots u_m), x_2, \cdots x_n; u_1, \cdots u_m) \\ &= \psi_i(x_2, \cdots x_n; u_1, \cdots u_m), && i = 1, \cdots n, \\ v_i &= u_i && i = 1, \cdots m. \end{aligned} \tag{4}$$

Here it is to be noted that the functions ψ_i, which are obviously of class C', are defined in a *complete* neighborhood of $(a_2, \cdots a_n; c_1, \cdots c_m)$. Moreover, their Jacobian at this point with respect to $x_2, \cdots x_n$, and u_1, is not zero. For if u_1 be called x_{n+1}, we have

$$\frac{\partial \psi_i}{\partial x_j} = \frac{\partial F_i}{\partial x_1}\frac{\partial \omega}{\partial x_j} + \frac{\partial F_i}{\partial x_j} = \frac{\partial F_i}{\partial x_1}\left(-\frac{\dfrac{\partial \Omega}{\partial x_j}}{\dfrac{\partial \Omega}{\partial x_1}}\right) + \frac{\partial F_i}{\partial x_j} \qquad \begin{cases} i = 1, \cdots n, \\ j = 2, \cdots n+1, \end{cases}$$

and this is precisely the form taken by the typical element in the minor of $\partial\Omega/\partial x_1$ in the determinant K after all the remaining elements of the last row have been made zero by the usual process. Since this determinant is not zero, the ordinary theorem on implicit functions tells us that there is a solution of the first n equations of (4) in the form

$$\begin{aligned} u_1 &= \chi_1(y_1, \cdots y_n; u_2, \cdots u_m), \\ x_i &= \chi_i(y_1, \cdots y_n; u_2, \cdots u_m), && i = 2, \cdots n, \end{aligned} \tag{5}$$

where the χ's are of class C'. If in (5), the first of these equations, we replace the u's by v's ,we have the equation

$$v_1 = \chi_1(y_1, \cdots y_n; v_2, \cdots v_m),$$

which represents the boundary of S'; that is to say, U^{-1} is defined in the vicinity of $(0, \cdots 0; c_1, \cdots c_m)$ in which $v_1 \geqq (\leqq)\chi_1$, and not in that where $v_1 \leqq (\geqq)\chi_1$.

This will be true in particular in the domain Σ' formed by setting $y_1 = y_2 = \cdots = y_n = 0$, the boundary then being

$$\begin{aligned} v_1 &= \chi_1(0, \cdots 0; v_2, \cdots v_m) \\ &= \lambda(v_1, \cdots v_m). \end{aligned}$$

Now Σ' is carried by the transformation U^{-1} into a region Σ of which the boundary is $u_1 = \lambda(u_2, \cdots u_n)$. This, if written in the form $\theta(u_1, \cdots u_m) = 0$, will satisfy the conditions of the theorem.

The treatment is very little changed in the excepted case where $\partial\Omega/\partial u_1$ is the only non-vanishing element in the last row of K. Here we solve equation (3) for u_1 and the equations corresponding to (4) for $x_1, \cdots x_n$, and substitute the values of the latter in the expression for u_1. This leads to an equation having exactly the same properties as (5). The reasoning and remaining details are exactly the same as before; and the proof is hereby completed.

If the matrix mentioned in this theorem is of rank n we may expect the boundary of Σ to have a singularity at the point in question. The character of this singularity may be determined—at least formally—by substituting in Ω the values of the x's and u's from U^{-1}, provided the necessary higher partial derivatives of Ω and the F's exist and are continuous.

6. Conclusion. The principal results of this article may be summed up in the following statement:

If the given functions are of class C', the ordinary theorems on implicit functions can be extended to the case where the initial solution is a boundary point, with only such changes as are obvious, *provided the region in the neighborhood of this boundary point is uniformly connected.*

Princeton University,
August, 1913.

A FORMULA IN THE THEORY OF SURFACES.

By R. D. Beetle.

The normal curvature $1/R$ and the geodesic torsion $1/T$ of a curve C lying on a surface S, whose first and second fundamental forms are $Edu^2 + 2Fdudv + Gdv^2$ and $Ddu^2 + 2D'dudv + D''dv^2$, are given by

$$(1) \qquad \frac{1}{R} = \frac{Ddu^2 + 2D'dudv + D''dv^2}{Edu^2 + 2Fdudv + Gdv^2},$$

and

$$(2) \qquad \frac{1}{T} = \frac{(FD - ED')du^2 + (GD - ED'')dudv + (GD' - FD'')dv^2}{H(Edu^2 + 2Fdudv + Gdv^2)},$$

where $H = \sqrt{EG - F^2}$. We may also write

$$(3) \qquad \frac{1}{T} = \frac{1}{\tau} - \frac{d\omega}{ds},$$

where $1/\tau$ is the torsion of the curve C, s is its arc, and ω is the angle which the normal to S makes with the principal normal of C.

In 1760, Euler* proved the relation

$$(4) \qquad \frac{1}{R} = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) + \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\cos 2\theta,$$

where ρ_1 and ρ_2 are the principal radii of normal curvature, and θ is the angle between the directions whose radii of normal curvature are R and ρ_1. In 1848, Bonnet† derived the corresponding formula for $1/T$,

$$(5) \qquad \frac{1}{T} = \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin 2\theta.$$

Introducing the total curvature

$$(6) \qquad K = \frac{DD'' - D'^2}{H^2} = \frac{1}{\rho_1\rho_2},$$

and the mean curvature

$$(7) \qquad K_m = \frac{ED'' + GD - 2FD'}{H^2} = \frac{1}{\rho_1} + \frac{1}{\rho_2},$$

we express (4) and (5) in the form

* L. Euler, Recherches sur la courbure des surfaces, Hist. de l'Acad. de Berlin, 16, 1760, p. 119.

† O. Bonnet, Memoire sur la theorie des surfaces, Journal de l'École Polytechnique, XXXII^e Cahier, p. 1; 1848.

$$(8) \qquad \frac{1}{R} = \frac{1}{2}(K_m + \sqrt{K_m^2 - 4K}\cos 2\theta),$$

and

$$(9) \qquad \frac{1}{T} = \frac{1}{2}\sqrt{K_m^2 - 4K}\sin 2\theta.$$

Combining (8) and (9), we obtain the formula

$$(10) \qquad \frac{1}{R^2} + \frac{1}{T^2} = \frac{K_m}{R} - K,$$

which, so far as I know, has hitherto escaped notice. It is easily derived directly from (1), (2), (6) and (7) by using as parametric curves an orthogonal system with the curve C one of the curves $v =$ const.

The relation (10) can be given the following geometrical interpretation. For each point of S, the locus of the point whose Cartesian coördinates are $(1/R, 1/T)$ is a circle with center M at $(\frac{1}{2}K_m, 0)$ and radius $\frac{1}{2}\sqrt{K_m^2 - 4K}$. The circle meets the axis of $1/R$ in the points $A_1 = (1/\rho_1, 0)$ and $A_2 = (1/\rho_2, 0)$, corresponding to the principal directions on S. It meets the axis of $1/T$ in the points $B_1 = (0, \sqrt{-K})$ and $B_2 = (0, -\sqrt{-K})$, corresponding to the asymptotic directions on S. From (4) and (5), it follows that, for every point P on the circle, $\angle PMA_1 = 2\theta$. To orthogonal directions on the surface correspond, therefore, diametrically opposite points on the circle.

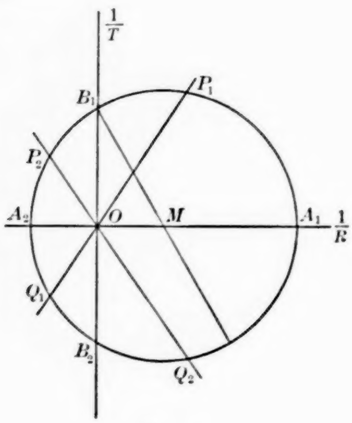


FIG. 1.

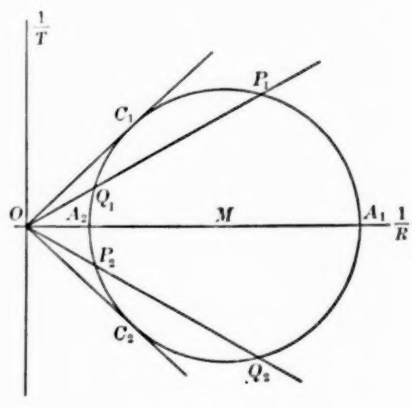


FIG. 2.

It is known, and easily verified directly from (1) and (2), that, if $(1/R_1, 1/T_1)$ and $(1/R_2, 1/T_2)$ correspond to conjugate directions, then $R_1/T_1 = -(R_2/T_2)$. It follows that, if P_1 and P_2 are the corresponding points on the circle, and O is the origin, the lines OP_1 and OP_2 are symmetrical with

respect to the coördinate axes. The lines OP_1 and OP_2 meet the circle in two other points, Q_1 and Q_2. The directions corresponding to P_1 and Q_2, or to P_2 and Q_1, are symmetrical with respect to the principal directions. The directions corresponding to Q_1 and Q_2 are conjugate. The directions corresponding to P_1 and Q_1, or to P_2 and Q_2, are in the relation called inverse-conjugate by Voss.*

Conversely, any two lines through O, and symmetrical with respect to the coördinate axes, meet the circle in four points which have the mutual relations just stated. To the points of contact,

$$C_1 = \left(\frac{2K}{K_m}, \frac{1}{K_m}\sqrt{K(K_m^2 - 4K)}\right),$$

$$C_2 = \left(\frac{2K}{K_m}, -\frac{1}{K_m}\sqrt{K(K_m^2 - 4K)}\right),$$

of the two tangents of the circle which pass through O, correspond the characteristic lines, the only self-inverse-conjugate lines.

From the geometric properties of the figure, we obtain the following theorems. With a few exceptions, they are well-known.

1. *On surfaces of negative total curvature (fig. 1), the asymptotic lines are real, but the characteristic lines are not. On surfaces of positive total curvature (fig. 2), the characteristic lines are real, but the asymptotic lines are not.*

2. *Each of the following is a characteristic property of minimal surfaces:*

(*a*) *the asymptotic lines form an orthogonal system;*

(*b*) *at each point, the value of $1/R^2 + 1/T^2$ is the same for all directions, being equal to minus the total curvature of the surface;*

(*c*) *at each point, the value of $1/\rho^2 + 1/\tau^2$ is the same for all the geodesics through the point, that is, the square of the total curvature of every geodesic is equal to the negative of the total curvature of the surface.*

For each is a necessary and sufficient condition that M fall on O.

3. *In order that a non-minimal surface be developable, it is necessary and sufficient that, at each point, the two asymptotic directions coincide.*

For the circle is then tangent at O to the axis of $1/T$.

4. *On developable minimal surfaces, that is, on planes, both asymptotic lines and lines of curvature are undetermined.*

The circle reduces to a point, the origin O.

5. *On spheres, the lines of curvature are undetermined, but the asymptotic lines are not.*

The circle reduces to the point M, distinct from O.

6. *The geodesic torsions in two orthogonal directions differ only in sign.*

* A. Voss, Über diejenigen Flächen, auf denen zwei Schaaren geodätischer Linien ein conjugirtes System bilden, Sitzungsberichte der K. Akademie zu München, vol. 18 (1888), p. 96.

For the corresponding points, being at the extremities of a diameter of the circle, have ordinates which differ only in sign.

7. *The sum of the normal curvatures in two orthogonal directions is constant at each point of S, and equals the mean curvature K_m.*

For the sum of the abscissas of two diametrically opposite points of the circle is equal to twice the abscissa of the center of the circle.

8. *It is a characteristic property of the orthogonal trajectories of the asymptotic lines that their normal curvature is equal to the mean curvature of the surface.*

9. *For two asymptotic lines through a point, the torsion at the point differs only in sign. The square of the torsion equals minus the total curvature of the surface.*

10. *For two characteristic lines through a point, the normal curvature at the point is the same. The radius of normal curvature in the direction of a characteristic line is equal to one-half the sum of the principal radii of normal curvature. The geodesic torsion of the characteristic lines through a point differs at the point only in sign.*

11. *The tangent of the angle between the asymptotic directions on any surface is*

$$\pm \frac{2\sqrt{-K}}{K_m} = \pm \frac{2\sqrt{-\rho_1\rho_2}}{\rho_1 + \rho_2}.$$

12. *The tangent of the angle between the characteristic directions on any surface is*

$$\pm 2\sqrt{\frac{K}{K_m^2 - 4K}} = \pm \frac{2\sqrt{\rho_1\rho_2}}{\rho_1 - \rho_2}.$$

13. *The geodesic torsion has its maximum and minimum values in the directions which bisect the principal directions.*

14. *For two inverse-conjugate directions, we have the relation $R_1/T_1 = R_2/T_2$. For two directions symmetric with respect to the lines of curvature, or for two conjugate directions, we have $R_1/T_1 = -(R_2/T_2)$.*

15. *On minimal surfaces, both for conjugate and inverse-conjugate directions, we have $R_1 + R_2 = 0$. In the first case $T_1 = T_2$. In the second case $T_1 = -T_2$.*

16. *For two conjugate or inverse-conjugate directions, the value of*

$$\left(\frac{1}{R_1^2} + \frac{1}{T_1^2}\right)\left(\frac{1}{R_2^2} + \frac{1}{T_2^2}\right)$$

is constant at a point, and equal to K^2.

17. *The square of the total curvature of a surface of Voss is equal to the product of the squares of the total curvatures of the conjugate geodesics.*

18. *For two conjugate or inverse-conjugate directions, the value of* $R_1 + R_2$ *is constant and equal to* $\rho_1 + \rho_2$.

This is an immediate consequence of 15, 16 and the relation (10).

19. *If* $R_1, R_2, \cdots, R_m$ *denote the values of* R *for* m *directions,* $m > 2$, *such that the angle between any two adjacent directions is* $2\pi/m$, *then*

$$\frac{1}{m}\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_m}\right) = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right).$$

For the corresponding points are situated at equal intervals on the circumference of the circle, and, from elementary mechanical principles, their average distance from the axis of $1/T$ must be equal to the distance of the center of the circle from that axis. Since this is equally true for any other line in the plane of the circle, we can state the more general theorem.

20. *If* $R_1, \cdots, R_m, T_1, \cdots, T_m$ *denote the values of* R *and* T *for* m *directions,* $m > 2$, *such that the angle between any two adjacent directions is* $2\pi/m$, *then*

$$\frac{1}{m}\sum_{i=1}^{m}\left(\frac{a}{R_i} + \frac{b}{T_i}\right) = \frac{a}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

for any constant values of a *and* b.

In particular, we have

$$\left(\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_m}\right) = 0.$$

By considering more complicated properties of the figure, still more general theorems can be derived.* We have adopted, throughout, the point of view of the theory of surfaces, but it should be noted that the method employed is equally available for the study of conic sections.

PRINCETON, N. J.,
January, 1914.

* See, for example, M. Chasles, Diverses propriétés des rayons vecteurs et des diamétres d'une section conique. Propriétés analogues des rayons de courbure des sections normales d'une surface, en un point, Comptes Rendus, 26, 1848.

THE DEVIATIONS OF FALLING BODIES.

By F. R. Moulton.*

1. Introduction. The subject of the deviations of freely falling bodies has been considered in many memoirs from the time of Laplace and Gauss to the recent work of Roever and Woodward.† All writers have agreed that a body falling from rest near the surface of the earth will deviate to the eastward with respect to a plumb-line hung from the initial point, but a great variety of results have been obtained regarding the deviation measured along the meridian. For example, Laplace found no meridional deviation, Gauss a small deviation toward the equator, Roever a deviation toward the equator several times that of Gauss, and Woodward a small deviation away from the equator. The diversity of these results is the occasion for this paper.

In the present paper the solution of the problem of the deviations of falling bodies is made to depend upon two well-established mathematical theories, viz., that of the solution of analytic differential equations in the neighborhood of regular points, and that of implicit analytic functions. The methods employed are rigorous, of general applicability, and relatively simple. The problem involves three parameters, viz., the height from which the body falls, the rate of rotation of the rotating body, and the oblateness of the rotating body. Only the first of these is arbitrary, and it is shown that the deviations of the falling body can be developed as converging power series in it if it is sufficiently small. In the meridional deviation the coefficient of the first power of this parameter, which is a function of the other two parameters and can be developed as a power series in them, is identically zero, at least in the case where the rotating body is a figure of revolution; but the coefficient of the second power of the parameter is not zero. The sign of this coefficient rigorously determines the character of the deviation when the distance through which the body falls is not too great. The principal part of the coefficient of the second power of the parameter agrees exactly with the result found by Roever in his first paper by an altogether different method. Differences in terms of higher order

* Research associate of the Carnegie Institution of Washington.

† For a more complete history of the question consult Roever's papers: Trans. of the Am. Math. Soc., vol. XII (1911), pp. 335–353, and vol. XIII (1912), pp. 469–490; and Woodward's articles: The Astron. Jour., vol. XXVIII (1913), pp. 17–29, and Science, vol. XXXVIII (1913), pp. 315–319.

would be found if they were computed because his definitions of the deviations differ somewhat from those used in this paper.

The question naturally arises why so great a variety of results has been obtained. The answer seems to lie in the fact that the solution of the problem depends upon two distinct mathematical processes, however much they may be concealed by the details of treatment; that both of them naturally lead to power series in all three parameters; that the coefficients of all terms of the first degree in the first parameter are identically zero; and that methods of approximation are not adapted to establishing identities. It is clear from what has been said that the correct result could never be established by explicitly developing the results in all three parameters because only a finite number of terms could be computed; and it is equally clear that great care would be necessary to secure exactly the same approximation in solving the differential equations and the implicit functions by approximating methods because the two processes are so different. Most writers have developed the results from the beginning in all three parameters, and have used approximations without proving their legitimacy.

2. The Differential Equations. Take the origin at the center of the rotating body and the principal polar axis coincident with the axis of rotation. Then, if r represents the distance, φ the latitude, $\theta = \theta_0 + \omega t + \lambda$ (where ω is the rate of rotation of the body) the longitude, and V the potential function, the differential equations of motion are

$$
(1)\qquad
\begin{aligned}
\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 - r\left(\omega + \frac{d\lambda}{dt}\right)^2\cos^2\varphi &= \frac{\partial V}{\partial r},\\
\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right) + r^2\left(\omega + \frac{d\lambda}{dt}\right)^2\sin\varphi\cos\varphi &= \frac{dV}{d\varphi},\\
\frac{d}{dt}\left[r^2\left(\omega + \frac{d\lambda}{dt}\right)\cos^2\varphi\right] &= \frac{\partial V}{\partial\lambda}.
\end{aligned}
$$

The potential function V in all cases can be developed as an infinite series of spherical harmonics whose coefficients depend upon the distribution of mass of the rotating body. If the rotating body is a figure of revolution about the axis of rotation whose density does not depend upon the longitude, the function V can be developed as a series of zonal harmonics of the form

$$
(2)\qquad V = \frac{\alpha}{r} + \frac{\beta}{r^3}(1 - 3\sin^2\varphi) + \cdots.
$$

Only the first two terms have sensible effects upon falling bodies and plumb-lines for short distances.

Let P_0 be the point from which the body falls and from which the plumb-line is suspended, $P_0^{(0)}$ the point where the line from the origin

to P_0 pierces the surface of the rotating body, P_1 the point where the falling body strikes the surface of the rotating body, and P_2 the point where the plumb-bob touches the surface of the rotating body. For notation

$$
(3) \quad
\begin{aligned}
&\text{let the coördinates of } P_0 \text{ be } r_0(1+h),\ \varphi_0,\ 0,\\
&\text{let the coördinates of } P_0^{(0)} \text{ be } r_0,\ \varphi_0,\ 0,\\
&\text{let the coördinates of } P_1 \text{ be } r_1,\ \varphi_1,\ \lambda_1,\\
&\text{let the coördinates of } P_2 \text{ be } r_2,\ \varphi_2,\ \lambda_2.
\end{aligned}
$$

The differential equations (1) are to be solved subject to the initial conditions

$$
(4) \quad
\begin{aligned}
r(0) &= r_0(1+h), & \varphi(0) &= \varphi_0, & \lambda(0) &= 0,\\
r'(0) &= 0, & \varphi'(0) &= 0, & \lambda'(0) &= 0.
\end{aligned}
$$

The problem is to find the points P_1 and P_2. If V is a sum of zonal harmonics, $\lambda_2 = 0$. When P_1 and P_2 have been found the longitudinal and meridional deviations in angular measure are respectively $\lambda_1 - \lambda_2$ and $\varphi_1 - \varphi_2$. If the first is positive the deviation is to the east, and if the second is positive the deviation is from the equator.

3. Integration of Equations (1). Since equations (1) are regular in the vicinity of the initial conditions, it follows from the general theory of analytic differential equations that they can be integrated uniquely as power series in t which converge for $|t|$ sufficiently small. It follows from (4) that the solution has no terms of the first degree in t. Hence the solution in the general case in which V is a sum of spherical harmonics has the form

$$
(5) \quad
\begin{aligned}
r &= r_0(1+h) + 0t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots,\\
\varphi &= \varphi_0 + 0t + b_2t^2 + b_3t^3 + b_4t^4 + \cdots,\\
\lambda &= 0 + 0t + c_2t^2 + c_3t^3 + c_4t^4 + \cdots,
\end{aligned}
$$

where the a_i, b_i, and c_i are constants which can be obtained by substituting (5) in (1) and equating coefficients of corresponding powers of t.

But if V is a sum of zonal harmonics, and consequently independent of λ, it follows from the third equation of (1) that $c_2 = 0$; and then from the first two equations of (1) that $a_3 = b_3 = 0$. Consequently, in this case the solution has the form

$$
(6) \quad
\begin{aligned}
r &= r_0(1+h) + a_2t^2 + 0t^3 + a_4t^4 + a_5t^5 + \cdots,\\
\varphi &= \varphi_0 + b_2t^2 + 0t^3 + b_4t^4 + b_5t^5 + \cdots,\\
\lambda &= 0 + 0 + c_3t^3 + c_4t^4 + c_5t^5 + \cdots.
\end{aligned}
$$

In general, the coefficients of all powers of t beyond the third are distinct from zero.

The full details will be given only in the case where V is a sum of zonal harmonics. It is found that

$$
\begin{aligned}
2a_2 &= \omega^2 r_0(1+h)\cos^2\varphi_0 + \left(\frac{\partial V}{\partial r}\right)_0, \\
2b_2 &= -\tfrac{1}{2}\omega^2 \sin 2\varphi_0 - \frac{1}{r_0^2(1+h)^2}\left(\frac{\partial V}{\partial \varphi}\right)_0, \\
6c_3 &= -4\omega\left[\frac{a_2}{r_0(1+h)} - b_2 \tan\varphi_0\right], \\
12a_4 &= 4r_0(1+h)b_2^2 + \omega^2\cos^2\varphi_0 a_2 - r_0(1+h)\omega^2 \sin 2\varphi_0 b_2 \\
&\qquad + 6r_0(1+h)\cos^2\varphi_0 c_3 + \left(\frac{\partial^2 V}{\partial r^2}\right)_0 a_2 + \left(\frac{\partial^2 V}{\partial r\partial\varphi}\right)_0 b_2, \\
12b_4 &= -\frac{12a_2b_2}{r_0(1+h)} - \frac{\omega^2 \sin 2\varphi_0}{r_0(1+h)} a_2 - 3\omega \sin 2\varphi_0 c_3 - \omega^2 \cos 2\varphi_0 b_2 \\
&\qquad + \left(\frac{\partial^2 V}{\partial r\partial\varphi}\right)_0 a_2 + \left(\frac{\partial^2 V}{\partial \varphi^2}\right)_0 b_2.
\end{aligned} \tag{7}
$$

The coefficients of higher powers of t are not needed.

4. Determination of the Coördinates of P_1. Let t_1 be the time at which the falling body reaches the surface of the rotating body. Then the coördinates of P_1 are obtained by replacing t by t_1 in equations (6). Hence the value of t_1 must be found.

The point P_1 is subject to the condition that it shall be on the surface of the rotating body, which is an equipotential surface for the potential function V and the rotational potential $\frac{1}{2}\omega^2 r^2 \cos^2\varphi$. Hence the coördinates of P_1 satisfy the equation

$$\tfrac{1}{2}\omega^2 r_1^2 \cos^2\varphi_1 + \frac{\alpha}{r_1} + \frac{\beta}{r_1^3}(1 - 3\sin^2\varphi_1) + \cdots = C,$$

where C is determined by any point on the surface. But the coördinates of $P_0^{(0)}$ also lie on the surface of the rotating body and satisfy an equation of the same form. The difference of these equations, which eliminates the constant C, is

$$
\begin{aligned}
F \equiv \tfrac{1}{2}\omega^2(r_1^2\cos^2\varphi_1 - r_0^2\cos^2\varphi_0) + \alpha\left(\frac{1}{r_1} - \frac{1}{r_0}\right) + \beta\left(\frac{1}{r_1^3} - \frac{1}{r_0^3}\right) \\
- 3\beta\left(\frac{\sin^2\varphi_1}{r_1^3} - \frac{\sin^2\varphi_0}{r_0^3}\right) + \cdots = 0.
\end{aligned} \tag{8}
$$

If t_1 is put in (6) in place of t, and if the series for r_1 and φ_1 are substituted in (8), the resulting expression can be developed as a power series

in t_1 and h which will converge provided $|t_1|$ is sufficiently small and $|h| < 1$. That is, (8) takes the form

$$F = p(h, t_1) = 0, \tag{9}$$

where p is a power series in h and t_1. Moreover, it is easily found that

$$p(0, 0) = 0, \quad \left(\frac{\partial p}{\partial t_1}\right)_0 = 0, \quad \left(\frac{\partial^2 p}{\partial t_1^2}\right)_0 = 4[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2], \tag{10}$$

where $a_2^{(0)}$ and $b_2^{(0)}$ are obtained from a_2 and b_2 by putting h equal to zero. It follows from the theory of implicit functions that (9) is solvable for t_1 as a power series in $\sqrt{h}$, vanishing for $h = 0$ and converging for $|h|$ sufficiently small, provided $4[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2]$ is distinct from zero. But this quantity is the square of the acceleration of gravity at the point $P_0^{(0)}$, and consequently is not zero. Therefore the solution of (9) has the form

$$t_1 = \sqrt{h}\, p_1(\sqrt{h}), \tag{11}$$

where $p_1(\sqrt{h})$ is a power series in $\sqrt{h}$.

It has been stated that it is sufficient to have the developments of series (6) to terms of the fourth degree inclusive in t. If the series are terminated at this point and V is a sum of zonal harmonics, F becomes a function of t_1^2 and h, and equation (8) can be solved for t_1^2 as a power series in h of the form

$$t_1^2 = \alpha_1 h + \alpha_2 h^2 + \cdots. \tag{12}$$

The first two terms of this series, which alone are required, are identical with the first two terms of the square of (11).

In order to find the coefficients α_1 and α_2 the explicit form of (9) is required. It is found from (6), (7), and (8) that

$$\begin{aligned}
F &= a_{10}t_1^2 + a_{01}h + a_{20}t_1^4 + a_{11}t_1^2h + a_{02}h^2 + \cdots = 0,\\
a_{10} &= 2[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2],\\
a_{01} &= 2r_0a_2^{(0)},\\
a_{20} &= 2a_2^{(0)}a_4^{(0)} + 2r_0^2b_2^{(0)}b_4^{(0)}\\
&\quad + \left[\tfrac{1}{2}\omega^2\cos^2\varphi_0 + \frac{\alpha}{r_0^3} + \frac{6\beta}{r_0^5}(1 - 3\sin^2\varphi_0)\right](a_2^{(0)})^2\\
&\quad - \left[r_0\omega^2 - \frac{9\beta}{r_0^4}\right]\sin 2\varphi_0 a_2^{(0)}b_2^{(0)} - \left[\tfrac{1}{2}r_0^2\omega^2 + \frac{3\beta}{r_0^3}\right]a_2^{(0)}b_2^{(0)},\\
a_{11} &= \left[r_0\omega^2\cos^2\varphi_0 + \frac{2\alpha}{r_0^2} + \frac{12\beta}{r_0^4}(1 - 3\sin^2\varphi_0) + 2\left(\frac{\partial a_2}{\partial h}\right)_0\right]a_2^{(0)}\\
&\quad + \left[-r_0^2\omega^2\sin 2\varphi_0 + \frac{9\beta}{r_0^3}\sin 2\varphi_0 + 2r_0^2\left(\frac{\partial b_2}{\partial h}\right)_0\right]b_2^{(0)},\\
a_{02} &= \tfrac{1}{2}r_0^2\omega^2\cos^2\varphi_0 + \frac{\alpha}{r_0} + \frac{6\beta}{r_0^3}(1 - 3\sin^2\varphi_0).
\end{aligned} \tag{13}$$

On substituting (12) in the first of (13) and equating coefficients of corresponding powers of h, it is found that

$$(14)\quad \alpha_1 = \frac{-r_0 a_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2}, \qquad \alpha_2 = -\frac{[a_{10}^2 a_{02} - a_{11}a_{01}a_{10} + a_{01}^2 a_{20}]}{a_{10}^3}.$$

On substituting (12) in the second and third equations of (6), it is found that

$$(15)\quad \begin{aligned} \varphi_1 &= \varphi_0 - \frac{r_0 a_2^{(0)} b_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2} h + \left[\alpha_1\left(\frac{\partial b_2}{\partial h}\right)_0 + \alpha_2 b_2^{(0)} \right. \\ &\qquad\qquad \left. + \alpha_1^2 b_4^{(0)}\right] h^2 + \cdots, \\ \lambda_1 &= 0 + \left\{\frac{-r_0 a_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2}\right\}^{\frac{3}{2}} c_3^{(0)} h^{\frac{3}{2}} + \cdots. \end{aligned}$$

5. Determination of the Coördinates of P_2. The point P_2 is determined by the conditions that it shall lie on the surface of the rotating body and that the normal to the equipotential surface passing through it shall also pass through P_0. The first of these conditions gives an equation which differs from (8) only in that the subscript 2 appears in place of 1.

Let

$$(16)\quad W = \tfrac{1}{2}\omega^2 r^2 \cos^2\varphi + V.$$

Then the equations of the normal to the equipotential surface $W = C$ at the point $P_2(x_2, y_2, z_2)$ are

$$(17)\quad \frac{\left(\frac{\partial W}{\partial x}\right)_2}{x - x_2} = \frac{\left(\frac{\partial W}{\partial y}\right)_2}{y - y_2} = \frac{\left(\frac{\partial W}{\partial z}\right)_2}{z - z_2}.$$

If this line passes through P_0 these equations are satisfied by

$$(18)\quad \begin{aligned} x &= x_0 = r_0(1 + h)\cos\varphi_0 \cos\theta_0, \\ y &= y_0 = r_0(1 + h)\cos\varphi_0 \sin\theta_0, \\ z &= z_0 = r_0(1 + h)\sin\varphi_0. \end{aligned}$$

The two equations (17) and the one which expresses the condition that P_2 shall be on the surface of the rotating body are sufficient to determine x_2, y_2, and z_2, or preferably their equivalents in polar coördinates.

Suppose V is a sum of zonal harmonics. Then the two equations (17) can be replaced by a single one, which is, if θ_0 is taken equal to zero,

$$(z_0 - z_2)\left(\frac{\partial W}{\partial x}\right)_2 - (x_0 - x_2)\left(\frac{\partial W}{\partial z}\right)_2 = 0.$$

The equation which expresses the condition that P_2 shall be on the surface $W = C$ and this equation become respectively in polar coördinates, after some simplifications,

$$
\begin{aligned}
(19)\quad F_2 &\equiv \tfrac{1}{2}\omega^2(r_2^2\cos^2\varphi_2 - r_0^2\cos^2\varphi_0) + \alpha\left(\frac{1}{r_2} - \frac{1}{r_0}\right) \\
&\quad + \beta\left(\frac{1}{r_2^3} - \frac{1}{r_0^3}\right) - 3\beta\left(\frac{\sin^2\varphi_2}{r_2^3} - \frac{\sin^2\varphi_0}{r_0^3}\right) = 0, \\
G_2 &\equiv \frac{r_0(1+h)}{r_2^2}\left[\alpha + \frac{3\beta}{r_2^2}(1 - 5\sin^2\varphi_2)\right]\sin(\varphi_2 - \varphi_0) \\
&\quad + \frac{6\beta}{r_2^4}[r_0(1+h)\cos\varphi_0 - r_2\cos\varphi_2]\sin\varphi_2 \\
&\quad + r_2\omega^2[r_0(1+h)\sin\varphi_0 - r_2\sin\varphi_2]\cos\varphi_2 = 0,
\end{aligned}
$$

from which r_2 and φ_2 can be determined.

In order to solve (19) it is convenient to let

$$
(20)\qquad r_2 = r_0(1+\rho), \quad \varphi_2 = \varphi_0 + \sigma.
$$

Then it follows that F_2 and G_2 can be expanded as power series in ρ, σ, and h of the form

$$
(21)\qquad F_2 = f_2(\rho, \sigma, h) = 0, \quad G_2 = g_2(\rho, \sigma, h) = 0.
$$

It is noted that f_2 does not involve h, g_2 contains h linearly, and both f_2 and g_2 converge for any finite values of σ and h provided $|\rho| < 1$.

Both of equations (21) are satisfied by $\rho = \sigma = h = 0$. The condition that they shall be uniquely solvable for ρ and σ as power series in h, vanishing for $h = 0$, is that the Jacobian of f_2 and g_2 with respect to ρ and σ shall be distinct from zero for $\rho = \sigma = h = 0$. It is found from (19) that

$$
(22)\qquad J = \begin{vmatrix} \dfrac{\partial f_2}{\partial \rho}, & \dfrac{\partial f_2}{\partial \sigma} \\ \dfrac{\partial g_2}{\partial \rho}, & \dfrac{\partial g_2}{\partial \sigma} \end{vmatrix} = \begin{vmatrix} 2r_0a_2^{(0)}, & +2r_0^2b_2^{(0)} \\ 2r_0^2b_2^{(0)}, & -2r_0a_2^{(0)} \end{vmatrix} = -4r_0^2[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2],
$$

which is distinct from zero because it is $-r_0^2$ times the square of the acceleration of gravity at $P_0^{(0)}$. Therefore the solution of (21) has the form

$$
(23)\qquad \rho = \beta_1 h + \beta_2 h^2 + \cdots, \quad \sigma = \gamma_1 h + \gamma_2 h^2 + \cdots,
$$

which converge provided $|h|$ is sufficiently small.

In order to determine the coefficients of (23) it is necessary to have the explicit expansions of (21). On referring to (19) it is seen that these series have the form

$$F_2 = b_{100}\rho + b_{010}\sigma + b_{200}\rho^2 + b_{110}\rho\sigma + b_{020}\sigma^2 + \cdots = 0,$$

$$(24)\qquad G_2 = c_{100}\rho + c_{010}\sigma + c_{001}h + c_{200}\rho^2 + c_{110}\rho\sigma + c_{101}\rho h + c_{020}\sigma^2 + c_{011}\sigma h + \cdots = 0,$$

where all terms which are not zero up to the second order in ρ, σ, and h have been written. It is easily found from equations (19) that the explicit values of the coefficients of (24) are

$$(25)\qquad \begin{aligned}
b_{100} &= 2r_0a_2^{(0)}, \qquad b_{010} = 2r_0^2b_2^{(0)},\\
b_{200} &= \tfrac{1}{2}r_0^2\omega^2\cos^2\varphi_0 + \frac{\alpha}{r_0} + \frac{6\beta}{r_0^3}(1 - 3\sin^2\varphi_0),\\
b_{110} &= \left[-r_0^2\omega^2 + \frac{9\beta}{r_0^3}\right]\sin 2\varphi_0,\\
b_{020} &= -\left[\tfrac{1}{2}r_0^2\omega^2 + \frac{3\beta}{r_0^3}\right]\cos 2\varphi_0,\\
c_{100} &= 2r_0^2b_2^{(0)}, \qquad c_{010} = -2r_0a_2^{(0)},\\
c_{001} &= -2r_0^2b_2^{(0)}, \qquad c_{200} = \left[-\tfrac{1}{2}r_0^2\omega^2 + \frac{12\beta}{r_0^3}\right]\sin 2\varphi_0,\\
c_{110} &= -r_0^2\omega^2(2 - 3\sin^2\varphi_0) - \frac{2\alpha}{r_0} - \frac{6\beta}{r_0^3}(3 - 8\sin^2\varphi_0),\\
c_{101} &= \left[\tfrac{1}{2}r_0^2\omega^2 - \frac{12\beta}{r_0^3}\right]\sin 2\varphi_0,\\
c_{020} &= \left[\tfrac{3}{4}r_0^2\omega^2 - \frac{21}{2}\frac{\beta}{r_0^3}\right]\sin 2\varphi_0,\\
c_{011} &= -r_0^2\omega^2\sin^2\varphi_0 + \frac{\alpha}{r_0} + \frac{3\beta}{r_0^3}(3 - 7\sin^2\varphi_0).
\end{aligned}$$

In order to obtain the meridional deviation it is sufficient to compute γ_1 and γ_2, but in order to obtain γ_2 it is necessary to compute β_1. It is found on substituting equations (23) in (24) and equating to zero the coefficients of corresponding powers of h, and then solving the resulting linear equations, that

$$(26)\qquad \begin{aligned}
\beta_1 &= \frac{r_0^2(b_2^{(0)})^2}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2},\\
\gamma_1 &= \frac{-r_0a_2^{(0)}b_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2},\\
D\gamma_2 &= -b_{100}c_{011}\gamma_1 - (b_{100}c_{200} - c_{100}b_{200})\beta_1^2\\
&\quad - (b_{100}c_{110} - c_{100}b_{110})\beta_1\gamma_1 - (b_{100}c_{020} - c_{100}b_{020})\gamma_1^2 - b_{100}c_{101}\beta_1,
\end{aligned}$$

where

$$D = -4r_0^2[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2]. \tag{27}$$

These results substituted in (23) and (20) give the coördinates of P_2 when V is a sum of zonal harmonics.

6. Expressions for the Deviations. In case V is a sum of zonal harmonics λ_2 is zero, and it follows from the second equation of (15) that the expression for the angular deviation in longitude is

$$\lambda_1 = \left\{ \frac{-r_0 a_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2} \right\}^{\frac{3}{2}} c_3^{(0)} h^{\frac{3}{2}} + \cdots. \tag{28}$$

The value of h, which is entirely arbitrary, can be taken so small that the sign of λ will be determined by the first term of this series.

It follows from (7) and (2) that

$$\begin{aligned} a_2^{(0)} &= -\frac{\alpha}{2r_0^2} + \tfrac{1}{2} r_0 \omega^2 \cos^2 \varphi_0 - \frac{3\beta}{2r_0^4}(1 - 3\sin^2 \varphi_0), \\ b_2^{(0)} &= -\tfrac{1}{4}\omega^2 \sin 2\varphi_0 - \frac{3\beta}{2r_0^5} \sin 2\varphi_0. \end{aligned} \tag{29}$$

The quantity $a_2^{(0)}$ is always negative for an actual physical body for otherwise there would be an acceleration outward along the radius at $P_0^{(0)}$. In the case of the earth the first term of the expression for $a_2^{(0)}$ is numerically about 300 times the coefficient of $\cos^2 \varphi_0$ in the second term, and about 600 times the coefficient of $(1 - 3\sin^2 \varphi_0)$ in the third term. That is,

$$\frac{\alpha}{r_0^2} = 300 r_0 \omega^2 = 600 \frac{3\beta}{r_0^4} \tag{30}$$

approximately. Consequently

$$\begin{aligned} c_3^{(0)} &= -\tfrac{2}{3}\omega \left[\frac{a_2^{(0)}}{r_0} - b_2^{(0)} \tan \varphi_0 \right] \\ &= \tfrac{2}{3}\omega \left[\frac{\alpha}{2r_0^3} - \tfrac{1}{2}\omega^2 + \frac{3\beta}{2r_0^5}(1 - 5\sin^2 \varphi_0) \right] \end{aligned} \tag{31}$$

is positive. Therefore the deviation is toward the eastward. Since $b_2^{(0)}$ is small compared to $a_2^{(0)}$, an approximate expression for the easterly deviation, obtained by neglecting $b_2^{(0)}$ in the denominator of (28) and the terms in $a_2^{(0)}$ and $c_3^{(0)}$ which are multiplied by ω^2 or β, is

$$\lambda_1 = \left(\frac{2\sqrt{2}}{3} \frac{r_0^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}} \omega + \cdots \right) h^{\frac{3}{2}} + \cdots. \tag{32}$$

By making use of (23) and the explicit value of γ_1 given in (26), the meridional deviation is found from (15) and (20) to be

$$(33)\quad \varphi_1 - \varphi_2 = 0h + \left[\alpha_1\left(\frac{\partial b_2}{\partial h}\right)_0 + \alpha_2 b_2^{(0)} + \alpha_1^2 b_4^{(0)} - \gamma_2\right]h^2 + \cdots.$$

The first important fact to be noted is that the coefficient of h is identically zero. It follows from the first equation of (15) and from (29) and (30) that the numerator and denominator of the coefficient of h in the expression for φ_1 are polynomials in ω^2 and β, and that the quotient can be expanded as an infinite converging power series in ω^2 and β. Exactly the same is of course true for the coefficient of h in the expression for φ_2. The quantities φ_1 and φ_2 are derived by quite different processes, the former involving the solution of differential equations and the latter only the solution of implicit functions. Hence it is clear that methods of approximation in carrying out these different processes are beset with danger because the effects upon the results are apt not to be the same in both.

The coefficient of h^2 in (33) is exact, but in order to determine its sign all terms except those of lowest order in ω^2 and β may be neglected. It follows from the numerical coefficients which are involved and the relations given in (30) that, at least for the earth, these simplifications can not change the sign of the result. It follows from (29) that $a_2^{(0)}$ is of order zero in ω^2 and β, and that $b_2^{(0)}$ is of order one in these same quantities. Then it is seen from (7) that $b_4^{(0)}$ and $(\partial b_2/\partial h)_0$ are of order one in ω^2 and β, though most of the terms in $b_4^{(0)}$ are of higher order. They are explicitly

$$(34)\quad \begin{aligned} b_4^{(0)} &= -\frac{\alpha}{24r_0^3}\left(4\omega^2 + \frac{27\beta}{r_0^5}\right)\sin 2\varphi_0 + \text{terms of the second order,}\\ \left(\frac{\partial b_2}{\partial h}\right)_0 &= +\frac{15\beta}{2r_0^5}\sin 2\varphi_0. \end{aligned}$$

It follows from (14) that α_1 and α_2 are of order zero in ω^2 and β, and from the first two equations of (26) that β_1 is of the second order and γ_1 is of the first order.

Now consider γ_2. Since D is of order zero all the terms in the right member of the third equation of (26), except possibly the first, are of the second order at least. It is seen from the first and last equations of (25) that b_{100} and c_{011} are of order zero. Therefore the first term of γ_2 is of order one.

On retaining only the terms of the first order in the coefficient of h^2 in (33), it is found that the approximate expression for the meridional deviation is

$$(35) \qquad \varphi_1 - \varphi_2 = \left[\frac{1}{2}\frac{r_0^3}{\alpha}\left(\frac{5\beta}{r_0^5} - 4\omega^2\right)\sin 2\varphi_0 + \cdots\right]h^2 + \cdots.$$

For the earth $4\omega^2$ is more than five times as great as $5\beta/r_0^5$. Therefore *the deviation of a body freely falling a small distance near the earth's surface is equatorward for all latitudes between* 0 *and* $\pm$ 90°.

THE UNIVERSITY OF CHICAGO,
December 1, 1913.

PROPERTIES OF CERTAIN HOMOGENEOUS LINEAR SUBSTITUTIONS.

By Harold Hilton.

1. Professor Loewy has discussed* the properties of a homogeneous linear substitution A

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \cdots + a_{tm}x_m \quad (t = 1, 2, \cdots, m),$$

which has as an invariant

$$\sum_t \epsilon_{tk} x_t \bar{x}_t \equiv x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_k\bar{x}_k - x_{k+1}\bar{x}_{k+1} - \cdots - x_m\bar{x}_m,$$

where x, $\bar{x}$ denote conjugate complex quantities, and $\epsilon_{ij} = 1$ when i, j are both $> k$ or both $\leq k$, and $\epsilon_{ij} = -1$ when one of i and j is $> k$ and the other is $\leq k$ $(m \geq k \geq 0)$.

I shall call the substitution A *quasi-unitary* in this case ("unitary" if $k = m$).

If A, instead of being quasi-unitary, satisfies the conditions $a_{ij} = \epsilon_{ij}\bar{a}_{ji}$ A will be called *quasi-Hermitian* ("Hermitian" if $k = m$). For instance, the substitution with matrix

$$\begin{vmatrix} a & w & g & p \\ \bar{h} & b & f & q \\ \bar{g} & \bar{f} & c & r \\ -\bar{p} & -\bar{q} & -\bar{r} & s \end{vmatrix}$$

is quasi-Hermitian, if a, b, c, s are real $(k = 3,\ m = 4)$.

The main interest of a quasi-Hermitian substitution lies in the fact that it bears a relation to a quasi-unitary substitution similar to that borne by a symmetric substitution to an orthogonal substitution. For instance, a quasi-Hermitian substitution is transformed by a quasi-unitary substitution into a quasi-Hermitian substitution, just as a symmetric substitution is transformed by an orthogonal substitution into a symmetric substitution;† and similar relations are developed in §§ 4 and 5.

But a quasi-Hermitian substitution has also properties analogous to important properties of a quasi-unitary substitution, as proved in §§ 2 and 3.

* Math. Annalen, 50 (1898), pp. 563, 564.

† Proc. London Math. Soc., 2, X (1911)., p. 274.

2. Professor Loewy (loc. cit.) has pointed out that those invariant-factors (elementartheiler) of a quasi-unitary substitution, which are not of the form $(\lambda - \alpha)^a$ where $\alpha\bar{\alpha} = 1$, can be grouped into pairs of the type $(\lambda - \alpha)^a$, $(\lambda - \bar{\alpha}^{-1})^a$ where $\alpha\bar{\alpha} \neq 1$.* We have similarly:—*Those invariant-factors of a quasi-Hermitian substitution which are not of the form* $(\lambda - \alpha)^a$ *where* α *is real, can be grouped into pairs of the type* $(\lambda - \alpha)^a$, $(\lambda - \bar{\alpha})^a$.

For suppose that A is the quasi-Hermitian substitution

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \cdots + a_{tm}x_m \qquad (t = 1, 2, \cdots, m),$$

transformed by P^{-1} into the canonical substitution N; so that $P^{-1}NP = A$. By a "canonical substitution" we mean a substitution of the type

$$x_t' = \lambda_t x_t + \beta_t x_{t+1} \qquad (t = 1, 2, \cdots, m),$$

where β_t is 0 or 1, and is certainly 0 if $\lambda_t \neq \lambda_{t+1}$. It is well known that every substitution is transformable into such a canonical substitution.†

Suppose that C is the Hermitian substitution

$$x_t' = c_{t1}x_1 + c_{t2}x_2 + \cdots + c_{tm}x_m \qquad (t = 1, 2, \cdots, m),$$

where

$$c_{ij} = \bar{c}_{ji} = \sum_{t=1}^{m} \epsilon_{tk} p_{ti} \bar{p}_{tj}.$$

Then we have (Proc. London Math. Soc., 2, X (1912), p. 282)

$$\lambda_i c_{ij} + \beta_{i-1} c_{i-1\,j} = \bar{\lambda}_j c_{ij} + \beta_{j-1} c_{i\,j-1}.$$

From this it follows, as in Mess. Math. (1912), p. 148, that, if $\lambda_i \neq \bar{\lambda}_j$, then $c_{ij} = 0$; but that if $\lambda_i = \bar{\lambda}_j$, then

$$\begin{array}{lll} c_{i-1\,j} = 0 & \text{when} \quad \beta_{i-1} = 1 & \text{and} \quad \beta_{j-1} = 0, \\ c_{i\,j-1} = 0 & \text{when} \quad \beta_{i-1} = 0 & \text{and} \quad \beta_{j-1} = 1, \\ c_{i-1\,j} = c_{ij-1} & \text{when} \quad \beta_{i-1} = 1 & \text{and} \quad \beta_{j-1} = 1. \end{array}$$

For example, if N is

$$x_1' = \alpha x_1 + x_2, \quad x_2' = \alpha x_2 + x_3, \quad x_3' = \alpha x_3; \quad x_4' = \alpha x_4 + x_5,$$
$$x_5' = \alpha x_5; \quad x_6' = \alpha x_6 + x_7, \quad x_7' = \alpha x_7; \quad x_8' = \alpha x_8,$$

where α is real, C has a matrix of the type

* See also Proc. London Math. Soc., 2, XI (1912), p. 97.

† See, for instance, Mess. Math. (1909), p. 24.

$$\begin{vmatrix} 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & d & 0 & f & 0 \\ a & b & c & d & e & f & g & h \\ 0 & 0 & \bar{d} & 0 & i & 0 & k & 0 \\ 0 & \bar{d} & \bar{e} & i & j & k & l & m \\ 0 & 0 & \bar{f} & 0 & \bar{k} & 0 & n & 0 \\ 0 & \bar{f} & \bar{g} & \bar{k} & \bar{l} & n & p & q \\ 0 & 0 & \bar{h} & 0 & \bar{m} & 0 & \bar{q} & r \end{vmatrix},$$

a, b, c, i, j, n, p, r being real;
or again if N is

$$\left.\begin{array}{l} x_1' = \alpha x_1 + x_2, \quad x_2' = \alpha x_2; \quad x_3' = \alpha x_3; \\ x_4' = \bar{\alpha} x_4 + x_5, \quad x_5' = \bar{\alpha} x_5; \quad x_6' = \bar{\alpha} x_6 \end{array}\right\}$$

c has a matrix of the type

$$\begin{vmatrix} 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & a & b & d \\ 0 & 0 & 0 & 0 & f & e \\ 0 & \bar{a} & 0 & 0 & 0 & 0 \\ \bar{a} & \bar{b} & \bar{f} & 0 & 0 & 0 \\ 0 & \bar{d} & \bar{e} & 0 & 0 & 0 \end{vmatrix}.$$

From these examples the general form of C is clear.

Now $\Sigma c_{ij} x_i \bar{x}_j$ is what $\sum_t \epsilon_{tk} x_t \bar{x}_t$ becomes when A is transformed into N; and therefore the matrix of C considered as a determinant does not vanish. But this is readily seen to be impossible unless the complex invariant-factors of C are paired as stated in the above theorem.

3. The quasi-Hermitian substitution A when transformed into canonical form N becomes the direct product of substitutions of the form

$$\left.\begin{array}{l} x_1' = \alpha x_1 + x_2, \quad \cdots, \quad x_{s-1}' = \alpha x_{s-1} + x_s, \quad x_s' = \alpha x_s \\ y_1' = \bar{\alpha} y_1 + y_2, \quad \cdots, \quad y_{s-1}' = \bar{\alpha} y_{s-1} + y_s, \quad y_s' = \bar{\alpha} y_s \end{array}\right\}$$

where α is not real, and of substitutions of the form

$$x_1' = \alpha x_1 + x_2, \quad \cdots, \quad x_{s-1}' = \alpha x_{s-1} + x_s, \quad x_s' = \alpha x_s$$

where α is real.

Then we can choose the new variables x, y, X so that A becomes N and $\sum_t \epsilon_{tk} x_t \bar{x}_t$ becomes the sum of functions of the type

$$(x_1\bar{y}_s + \bar{x}_1 y_s) + (x_2\bar{y}_{s-1} + \bar{x}_2 y_{s-1}) + \cdots + (x_s\bar{y}_1 + \bar{x}_s y_1)$$

and

$$\pm (X_1\bar{X}_s + X_2\bar{X}_{s-1} + \cdots + X_{s-1}\bar{X}_2 + X_s\bar{X}_1)$$

respectively.

The proof of this statement is suppressed, since it is exactly similar to the proof of a similar theorem for symmetric substitutions published elsewhere.*

As in Proc. London Math. Soc., 2, XI (1911), pp. 98–100, we can show that when

$$(x_1\bar{y}_s + \bar{x}_1 y_s) + (x_2\bar{y}_{s-1} + \bar{x}_2 y_{s-1}) + \cdots + (x_s\bar{y}_1 + \bar{x}_s y_1)$$

is reduced by change of variables to the form

$$\pm z_1\bar{z}_1 \pm z_2\bar{z}_2 \pm z_3\bar{z}_3 \pm \cdots$$

there are s positive and s negative signs; while when

$$X_1\bar{X}_s + X_2\bar{X}_{s-1} + \cdots + X_s\bar{X}_1$$

is reduced to this form, the number of negative and the number of positive signs are equal when s is even, and differ by unity when s is odd. It follows that a series of properties of quasi-unitary substitutions established by Loewy (Math. Annalen, 50 (1898), pp. 563, 564), are also properties of a quasi-Hermitian substitution. For instance:—"The unreal characteristic-roots of the quasi-Hermitian substitution A are not more than $2k'$ in number; k' being the smaller of the quantities k and $m - k$. If exactly $m - 2k'$ characteristic-roots are real, they correspond to linear invariant-factors."

"A cannot have more than k' invariant-factors which are not linear. If A has exactly k' such invariant-factors they are all of degree 2 or 3. If they are all of degree 3, every characteristic-root of A is real."

4. A relation between quasi-Hermitian and quasi-unitary substitutions similar to that between symmetric and orthogonal substitutions, and proved in the same way,† is the following:—

All quasi-Hermitian substitutions with given invariant-factors can be transformed by a quasi-unitary substitution into the same quasi-Hermitian substitution which is the direct product of substitutions each of which has a single invariant-factor $(\lambda - \alpha)^a$ where α is real, or a pair of invariant-

* Proc. London Math. Soc., 2, XII (1913), p. 94.

† Mess. Math. (1912), p. 146; Bôcher's Higher Algebra, p. 302.

factors $(\lambda - \alpha)^a$, $(\lambda - \bar{\alpha})^a$. Similarly for quasi-unitary substitutions transformed by quasi-unitary substitutions.

Example:—Find a quasi-Hermitian substitution with invariant-factors

$$(\lambda - \alpha)^2, \qquad (\lambda - \bar{\alpha})^2.$$

Since

$$x_1\bar{x}_4 + x_2\bar{x}_3 + x_3\bar{x}_2 + x_4\bar{x}_1 \equiv (x + \tfrac{1}{2}x_4)(\bar{x}_1 + \tfrac{1}{2}\bar{x}_4) + (x_2 + \tfrac{1}{2}x_3)(\bar{x}_2 + \tfrac{1}{2}\bar{x}_3)$$
$$- (x_2 - \tfrac{1}{2}x_3)(\bar{x}_2 - \tfrac{1}{2}\bar{x}_3) - (x_1 - \tfrac{1}{2}x_4)(\bar{x}_1 - \tfrac{1}{2}\bar{x}_4)$$

the required substitution is $F^{-1}NF$, where N is the canonical substitution

$$x_1' = \alpha x_1 + x_2, \qquad x_2' = \alpha x_2; \qquad x_3' = \bar{\alpha} x_3 + x_4, \qquad x_4' = \bar{\alpha} x_4$$

and F is

$$x_1' = x_1 + \tfrac{1}{2}x_4, \qquad x_2' = x_2 + \tfrac{1}{2}x_3, \qquad x_3' = x_2 - \tfrac{1}{2}x_3, \qquad x_4' = x_1 - \tfrac{1}{2}x_4.$$

A similar process will find a quasi-Hermitian or symmetric substitution with any assigned invariant-factors.

5. If we are given any substitution A, there are substitutions permutable with A k-ply infinite in number, where k is known when the invariant-factors of A are given.* The problem suggests itself:—"To determine k if the substitutions permutable with A are limited in any way; if, for instance, they are orthogonal."

This problem can be solved in special cases. We have, for instance, the results:—

If a symmetric substitution A has α invariant-factors $(\lambda - \lambda_1)^a$, β invariant-factors $(\lambda - \lambda_1)^b$, γ invariant-factors $(\lambda - \lambda_1)^c$, $\cdots$, where

$$a > b > c > \cdots,$$

the orthogonal substitutions permutable with A are

$$\Sigma \tfrac{1}{2}\{\alpha(\alpha - 1)a + \beta(3\beta - 1)b + \gamma(5\gamma - 1)c + \delta(7\delta - 1)d + \cdots\}\text{-ply}$$

infinite in number; the summation being extended over each distinct characteristic-root of A.

If a quasi-Hermitian substitution A has ρ invariant-factors $(\lambda - \lambda_1)^r$ and $(\lambda - \bar{\lambda}_1)^r$, σ invariant-factors $(\lambda - \lambda_1)^s$ and $(\lambda - \bar{\lambda}_1)^s$, τ invariant-factors $(\lambda - \lambda_1)^t$ and $(\lambda - \bar{\lambda}_1)^t$, $\cdots$, $(r > s > t > \cdots)$, and α invariant-factors $(\lambda - \lambda_0)^a$, β invariant factors $(\lambda - \lambda_0)^b$, γ invariant-factors $(\lambda - \lambda_0)^c$, $\cdots$, $(a > b > c > \cdots)$, where λ_0 is real, the quasi-unitary substitutions permutable with A are

$$\{\Sigma[\rho^2 r + \sigma(2\rho + \sigma)s + \tau(2\rho + 2\sigma + \tau)t + \cdots]$$
$$+ \Sigma\tfrac{1}{2}[\alpha(\alpha - 1)a + \beta(3\beta - 1)b + \gamma(5\gamma - 1)c + \cdots]\}\text{-ply}$$

* Mess. Math. (1911), p. 112.

Then we can choose the new variables x, y, X so that A becomes N and $\sum_t \epsilon_{tk} x_t \bar{x}_t$ becomes the sum of functions of the type

$$(x_1\bar{y}_s + \bar{x}_1 y_s) + (x_2\bar{y}_{s-1} + \bar{x}_2 y_{s-1}) + \cdots + (x_s\bar{y}_1 + \bar{x}_s y_1)$$

and

$$\pm (X_1\bar{X}_s + X_2\bar{X}_{s-1} + \cdots + X_{s-1}\bar{X}_2 + X_s\bar{X}_1)$$

respectively.

The proof of this statement is suppressed, since it is exactly similar to the proof of a similar theorem for symmetric substitutions published elsewhere.*

As in Proc. London Math. Soc., 2, XI (1911), pp. 98–100, we can show that when

$$(x_1\bar{y}_s + \bar{x}_1 y_s) + (x_2\bar{y}_{s-1} + \bar{x}_2 y_{s-1}) + \cdots + (x_s\bar{y}_1 + \bar{x}_s y_1)$$

is reduced by change of variables to the form

$$\pm z_1\bar{z}_1 \pm z_2\bar{z}_2 \pm z_3\bar{z}_3 \pm \cdots$$

there are s positive and s negative signs; while when

$$X_1\bar{X}_s + X_2\bar{X}_{s-1} + \cdots + X_s\bar{X}_1$$

is reduced to this form, the number of negative and the number of positive signs are equal when s is even, and differ by unity when s is odd. It follows that a series of properties of quasi-unitary substitutions established by Loewy (Math. Annalen, 50 (1898), pp. 563, 564), are also properties of a quasi-Hermitian substitution. For instance:—"The unreal characteristic-roots of the quasi-Hermitian substitution A are not more than $2k'$ in number; k' being the smaller of the quantities k and $m - k$. If exactly $m - 2k'$ characteristic-roots are real, they correspond to linear invariant-factors."

"A cannot have more than k' invariant-factors which are not linear. If A has exactly k' such invariant-factors they are all of degree 2 or 3. If they are all of degree 3, every characteristic-root of A is real."

4. A relation between quasi-Hermitian and quasi-unitary substitutions similar to that between symmetric and orthogonal substitutions, and proved in the same way,† is the following:—

All quasi-Hermitian substitutions with given invariant-factors can be transformed by a quasi-unitary substitution into the same quasi-Hermitian substitution which is the direct product of substitutions each of which has a single invariant-factor $(\lambda - \alpha)^a$ where α is real, or a pair of invariant-

* Proc. London Math. Soc., 2, XII (1913), p. 94.

† Mess. Math. (1912), p. 146; Bôcher's Higher Algebra, p. 302.

factors $(\lambda - \alpha)^a$, $(\lambda - \bar{\alpha})^a$. Similarly for quasi-unitary substitutions transformed by quasi-unitary substitutions.

Example:—Find a quasi-Hermitian substitution with invariant-factors

$$(\lambda - \alpha)^2, \qquad (\lambda - \bar{\alpha})^2.$$

Since

$$x_1\bar{x}_4 + x_2\bar{x}_3 + x_3\bar{x}_2 + x_4\bar{x}_1 \equiv (x + \tfrac{1}{2}x_4)(\bar{x}_1 + \tfrac{1}{2}\bar{x}_4) + (x_2 + \tfrac{1}{2}x_3)(\bar{x}_2 + \tfrac{1}{2}\bar{x}_3)$$
$$- (x_2 - \tfrac{1}{2}x_3)(\bar{x}_2 - \tfrac{1}{2}\bar{x}_3) - (x_1 - \tfrac{1}{2}x_4)(\bar{x}_1 - \tfrac{1}{2}\bar{x}_4)$$

the required substitution is $F^{-1}NF$, where N is the canonical substitution

$$x_1' = \alpha x_1 + x_2, \qquad x_2' = \alpha x_2; \qquad x_3' = \bar{\alpha} x_3 + x_4, \qquad x_4' = \bar{\alpha} x_4$$

and F is

$$x_1' = x_1 + \tfrac{1}{2}x_4, \qquad x_2' = x_2 + \tfrac{1}{2}x_3, \qquad x_3' = x_2 - \tfrac{1}{2}x_3, \qquad x_4' = x_1 - \tfrac{1}{2}x_4.$$

A similar process will find a quasi-Hermitian or symmetric substitution with any assigned invariant-factors.

5. If we are given any substitution A, there are substitutions permutable with A k-ply infinite in number, where k is known when the invariant-factors of A are given.* The problem suggests itself:—"To determine k if the substitutions permutable with A are limited in any way; if, for instance, they are orthogonal."

This problem can be solved in special cases. We have, for instance, the results:—

If a symmetric substitution A has α invariant-factors $(\lambda - \lambda_1)^a$, β invariant-factors $(\lambda - \lambda_1)^b$, γ invariant-factors $(\lambda - \lambda_1)^c$, $\cdots$, where

$$a > b > c > \cdots,$$

the orthogonal substitutions permutable with A are

$$\Sigma\tfrac{1}{2}\{\alpha(\alpha - 1)a + \beta(3\beta - 1)b + \gamma(5\gamma - 1)c + \delta(7\delta - 1)d + \cdots\}\text{-ply}$$

infinite in number; the summation being extended over each distinct characteristic-root of A.

If a quasi-Hermitian substitution A has ρ invariant-factors $(\lambda - \lambda_1)^r$ and $(\lambda - \bar{\lambda}_1)^r$, σ invariant-factors $(\lambda - \lambda_1)^s$ and $(\lambda - \bar{\lambda}_1)^s$, τ invariant-factors $(\lambda - \lambda_1)^t$ and $(\lambda - \bar{\lambda}_1)^t$, $\cdots$, $(r > s > t > \cdots)$, and α invariant-factors $(\lambda - \lambda_0)^a$, β invariant factors $(\lambda - \lambda_0)^b$, γ invariant-factors $(\lambda - \lambda_0)^c$, $\cdots$, $(a > b > c > \cdots)$, where λ_0 is real, the quasi-unitary substitutions permutable with A are

$$\{\Sigma[\rho^2 r + \sigma(2\rho + \sigma)s + \tau(2\rho + 2\sigma + \tau)t + \cdots]$$
$$+ \Sigma\tfrac{1}{2}[\alpha(\alpha - 1)a + \beta(3\beta - 1)b + \gamma(5\gamma - 1)c + \cdots]\}\text{-ply}$$

* Mess. Math. (1911), p. 112.

infinite in number, the summations being extended over each pair of characteristic roots λ_1 and $\overline{\lambda_1}$ of A and over each real characteristic-root λ_0 of A.

I have obtained somewhat similar results for orthogonal substitutions permutable with a given orthogonal substitution, or quasi-unitary with quasi-unitary, in special cases; but the general result appears to be very complex.

The method of proof will be evident if we consider a particular example. Suppose, for instance, that the symmetric substitution A has invariant-factors $(\lambda - \alpha)^s$, $(\lambda - \alpha)^s$, $(\lambda - \alpha)^s$. It is required to prove that the orthogonal substitutions permutable with A are $\frac{1}{2}.3.2.s$-ply infinite in number.

Suppose that A, when transformed into canonical form $N = PAP^{-1}$, becomes

$$\left.\begin{array}{llll} x_1' = \alpha x_1 + x_2, & \cdots, & x_{s-1}' = \alpha x_{s-1} + x_s, & x_s' = \alpha x_s \\ \xi_1' = \alpha \xi_1 + \xi_2, & \cdots, & \xi_{s-1}' = \alpha \xi_{s-1} + \xi_s, & \xi_s' = \alpha \xi_s \\ X_1' = \alpha X_1 + X_2, & \cdots, & X_{s-1}' = \alpha X_{s-1} + X_s, & X_s' = \alpha X_s \end{array}\right\}.$$

Then if P is chosen properly,

$$x_1^2 + x_2^2 + x_3^2 + \cdots$$

becomes

$$\varphi_1(x, x) + \varphi_1(\xi, \xi) + \varphi_1(X, X)^*$$

where

$$\left.\begin{array}{ll} \varphi_1(x, y) \equiv & x_1 y_s + x_2 y_{s-1} + x_3 y_{s-2} + \cdots + x_{s-2} y_3 + x_{s-1} y_2 + x_s y_1 \\ \varphi_2(x, y) \equiv & x_2 y_s + x_3 y_{s-1} + \cdots + x_{s-1} y_3 + x_s y_2 \\ \varphi_3(x, y) \equiv & x_3 y_s + \cdots + x_s y_3 \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \varphi_s(x, y) \equiv & x_s y_s \end{array}\right\}.$$

Since if $BA = AB$,

$$PBP^{-1} \cdot PAP^{-1} = PAP^{-1} \cdot PBP^{-1};$$

the number of substitutions permutable with A is the same as the number permutable with N. It therefore suffices to prove that the substitutions permutable with N and having $\varphi_1(x, x) + \varphi_1(\xi, \xi) + \varphi_1(X, X)$ as an invariant are $\frac{1}{2}.3.2.s$-ply infinite in number.

* Proc. London Math. Soc., 2, XII (1913), p. 94.

Now the most general substitutions permutable with N will be

$$\left.\begin{aligned}
x_1' &= a_{11s}x_1 + a_{11\,s-1}x_2 + a_{11\,s-2}x_3 + \cdots + a_{111}x_s + a_{12s}\xi_1 + a_{12\,s-1}\xi_2\\
&\qquad + a_{12\,s-2}\xi_3 + \cdots + a_{121}\xi_s + a_{13s}X_1 + a_{13\,s-1}X_2 + a_{13\,s-2}X_3\\
&\qquad\qquad + \cdots + a_{131}X_s\\
x_2' &= a_{11s}x_2 + a_{11\,s-1}x_3 + \cdots + a_{112}x_s + a_{12s}\xi_2 + a_{12\,s-1}\xi_3 + \cdots\\
&\qquad + a_{122}\xi_s + a_{13s}X_2 + a_{13\,s-1}X_3 + \cdots + a_{132}X_s\\
x_3' &= a_{11s}x_3 + \cdots + a_{113}X_s + a_{12s}\xi_3 + \cdots + a_{123}\xi_s + a_{13s}X_3 + \cdots + a_{133}X_s\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
x_s' &= a_{11s}x_s + a_{12s}\xi_s + a_{13s}X_s,\\
\xi_1' &= a_{21s}x_1 + \cdots + a_{211}x_s + a_{22s}\xi_1 + \cdots + a_{221}\xi_s + a_{23s}X_1 + \cdots + a_{231}X_s\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
X_1' &= a_{31s}x_1 + \cdots + a_{311}x_s + a_{32s}\xi_1 + \cdots + a_{321}\xi_s + a_{33s}X_1 + \cdots + a_{331}X_s\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}\right\}.$$

Operating with this on $\varphi_1(x, x) + \varphi_1(\xi, \xi) + \varphi_1(X, X)$ we get*

$$\begin{aligned}
\sum_t \{ &[\varphi_s(a_{t1}, a_{t1}) \cdot \varphi_1(x, x) + \varphi_s(a_{t2}, a_{t1}) \cdot \varphi_1(\xi, x) + \varphi_s(a_{t3}, a_{t1}) \cdot \varphi_1(X, x)\\
&+ \varphi_s(a_{t1}, a_{t2})\varphi_1(x, \xi) + \varphi_s(a_{t2}, a_{t2}) \cdot \varphi_1(\xi, \xi) + \varphi_s(a_{t3}, a_{t2}) \cdot \varphi_1(X, \xi)\\
&+ \varphi_s(a_{t1}, a_{t3}) \cdot \varphi_1(x, X) + \varphi_s(a_{t2}, a_{t3}) \cdot \varphi_1(\xi, X) + \varphi_s(a_{t3}, a_{t3}) \cdot \varphi_1(X, X)]\\
&+ [\varphi_{s-1}(a_{t1}, a_{t1}) \cdot \varphi_2(x, x) + \cdots] + [\varphi_{s-2}(a_{t1}, a_{t1}) \cdot \varphi_3(x, x) + \cdots] + \cdots\}
\end{aligned}$$
$$(t = 1, 2, 3).$$

This must reduce to $\varphi_1(x, x) + \varphi_1(\xi, \xi) + \varphi_1(X, X)$.

Let $A_s, A_{s-1}, A_{s-2}, \cdots$ be the matrices whose general elements are respectively $a_{ij\,s}, a_{ij\,s-1}, a_{ij\,s-2}, \cdots$ $(i, j = 1, 2, 3)$.

Then we have in turn $A_sA_s' = 1$, $A_sA_{s-1}' + A_{s-1}A_s' = 0$, $A_sA_{s-2}' + A_{s-1}A_{s-1}' + A_{s-2}A_s' = 0$, $\cdots$.

Hence firstly, A is orthogonal so that the quantities a_{ijs} are functions of $\frac{1}{2}.3.2$ independent quantities.†

Then, when the quantities a_{ijs} are fixed, the quantities $a_{ij\,s-1}$ are functions of $\frac{1}{2}.3.2$ independent quantities, since $A_sA_{s-1}' + A_{s-1}A_s' = 0$.

In fact, given any non-singular matrix A of degree m, we can always find a matrix B of degree m, such that $AB' + BA'$ is a given symmetric matrix S, in a $\frac{1}{2}m(m-1)$-ply infinite number of ways. For take P any matrix whose elements p_{ij} are arbitrary if $j > i$, and whose elements p_{ij} are given by $S = P + P'$ when $i \leq j$. Then we may suppose $AB' = P$, $BA' = P'$; which gives B in terms of the $\frac{1}{2}m(m-1)$ arbitrary quantities p_{ij} $(j > i)$.

Then when the quantities a_{ijs}, $a_{ij\,s-1}$ are fixed, the quantities $a_{ij\,s-2}$ are functions of $\frac{1}{2}.3.2$ independent quantities, since $A_sA_{s-2}' + A_{s-1}A_{s-1}' + A_{s-2}A_s' = 0$. Continuing this process, we see that the required substitutions permutable with N are $\frac{1}{2}.3.2.s$-ply infinite in number.

---

* Proc. London Math. Soc., 2, XII (1913), p. 96

† Cayley, Crelle, XXXII (1846), p. 119, Collected Works, I, p. 332.

SEPTEMBER, 1913

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

ORMOND STONE	MAXIME BÔCHER
L. P. EISENHART	ELIJAH SWIFT
OSWALD VEBLEN	J. H. M. WEDDERBURN

PUBLISHED UNDER THE AUSPICES OF PRINCETON UNIVERSITY

SECOND SERIES, VOL. 15, NO. 1.

LANCASTER, PA., AND PRINCETON, N. J.

CONTENTS

ANNALS OF MATHEMATICS

Published in September, December, March and June, at Lancaster, Pa., and Princeton, N. J., U. S. A.

Communications should be addressed to The Editors of the *Annals of Mathematics*, Lancaster, Pa., or Princeton, N. J., U. S. A. Subscription price, $2 a volume (four numbers) in advance. Single numbers, 75 cents. All drafts and money orders should be made payable to THE ANNALS OF MATHEMATICS.

PRESS OF THE NEW ERA PRINTING COMPANY, LANCASTER, PA.

DECEMBER, 1913

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

PUBLISHED UNDER THE AUSPICES OF PRINCETON UNIVERSITY

SECOND SERIES, VOL. 15, NO. 2.

LANCASTER, PA., AND PRINCETON, N. J.

CONTENTS

ANNALS OF MATHEMATICS

Published in September, December, March and June, at Lancaster, Pa., and Princeton, N. J., U. S. A.

Communications should be addressed to The Editors of the *Annals of Mathematics*, Lancaster, Pa., or Princeton, N. J., U. S. A. Subscription price, $2 a volume (four numbers) in advance. Single numbers, 75 cents. All drafts and money orders should be made payable to The Annals of Mathematics.

Press of The New Era Printing Company, Lancaster, Pa.

Ma

MARCH, 1914

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

PUBLISHED UNDER THE AUSPICES OF PRINCETON UNIVERSITY

SECOND SERIES, VOL. 15, NO. 3.

LANCASTER, PA., AND PRINCETON, N. J.

CONTENTS

ANNALS OF MATHEMATICS

Published in September, December, March and June, at Lancaster, Pa., and Princeton, N. J., U. S. A.

Communications should be addressed to The Editors of the *Annals of Mathematics*, Lancaster, Pa., or Princeton, N. J., U. S. A. Subscription price, $2 a volume (four numbers) in advance. Single numbers, 75 cents. All drafts and money orders should be made payable to THE ANNALS OF MATHEMATICS.

PRESS OF THE NEW ERA PRINTING COMPANY, LANCASTER, PA.

JUNE, 1914

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

PUBLISHED UNDER THE AUSPICES OF PRINCETON UNIVERSITY

SECOND SERIES, VOL. 15, NO. 4.

LANCASTER, PA., AND PRINCETON, N. J.

CONTENTS

ANNALS OF MATHEMATICS

Published in September, December, March and June, at Lancaster, Pa., and Princeton, N. J., U. S. A.

Communications should be addressed to The Editors of the *Annals of Mathematics*, Lancaster, Pa., or Princeton, N. J., U. S. A. Subscription price, $2 a volume (four numbers) in advance. Single numbers, 75 cents. All drafts and money orders should be made payable to The Annals of Mathematics.

Press of The New Era Printing Company, Lancaster, Pa.